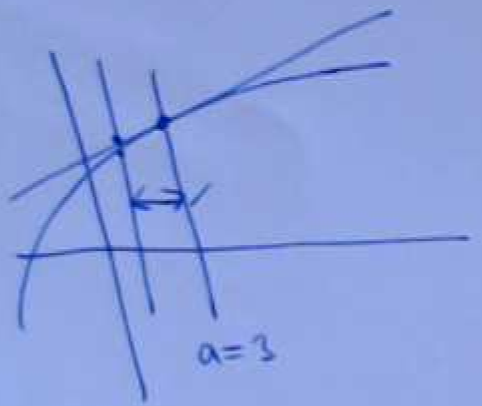
①

WW 4.1 Q3

$f(x) = \sqrt{x+33}$ $\quad a = 3 \quad \Delta x = -0.1$

$= (x+33)^{1/2}.$



$f(x+\Delta x) \approx f(x) + f'(x)\Delta x.$

$f'(x) = \frac{1}{2}(x+33)^{-1/2}.$

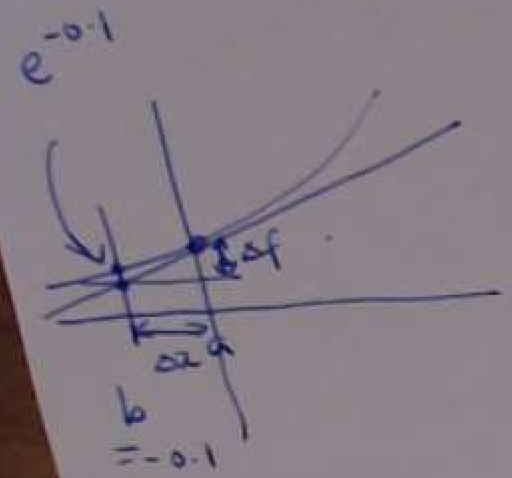
$f'(a=3) = \frac{1}{2}\big(\underbrace{3+33}_{36}\big)^{-1/2} = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12}.$

$f(2.9) \approx 6 - \frac{1}{12}\cdot 0.1 \qquad \Delta f \approx -\frac{1}{120}.$

$5\frac{119}{120}.$

error $= \left|\sqrt{2.9} - 5\frac{119}{120}\right|.$ $\quad$ percentage error. $\quad \dfrac{\left|\sqrt{2.9} - 5\frac{119}{120}\right|}{\sqrt{2.9}} \times 100.$

WW 4.1 Q7 $\quad f(x)=e^x \quad a=0 \quad b=-0.1$



$$f(x) \approx f(a) + f'(a)\,\Delta x.$$

$$f(a) = e^0 = 1.$$

$$f'(x) = e^x \qquad f'(0) = e^0 = 1.$$

$$\underbrace{f(-0.1)} \approx 1 + \underbrace{1 \cdot -0.1}_{\underbrace{\Delta f \approx -0.1}} = 0.9.$$

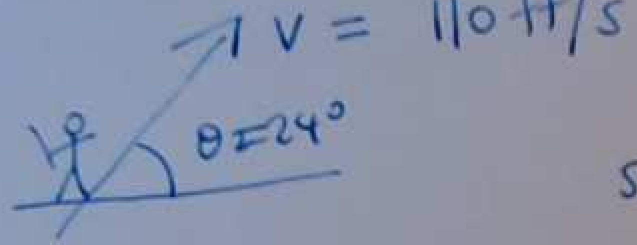
error $= |e^{-0.1} - 0.9|$

percentage error $\left|\dfrac{|e^{-0.1}-0.9|}{e^{-0.1}}\right| \times 100.$

approx value of $f(-0.1)$

approx change Δf from a to b.

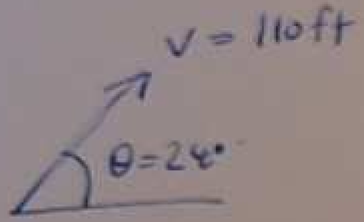
WW 4.1 Q10

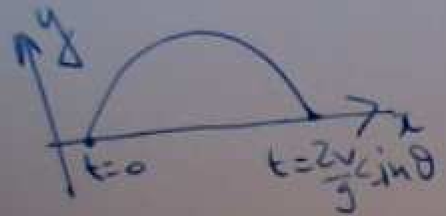


s = how far the golf ball travels.

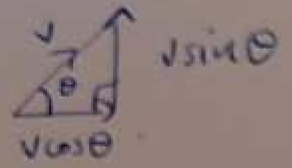
a) estimate Δs if $\Delta\theta = +3^\circ$.

b) estimate Δs if $\Delta v = +2$ ft/sec.





$g = 32\,ft/s^2$



↑ vsinθ

Motion in the vertical direction.

$$y(t) = v\sin\theta\, t - \tfrac{1}{2} g t^2$$

$$= t\left(v\sin\theta - \tfrac{1}{2} g t\right) = 0$$

$$t = \cancel{2gv\sin\theta}\ \ \frac{2v}{g}\sin\theta$$

$$x(t) = v\cos\theta\, t$$

$$s = v\cos\theta \cdot \frac{2v}{g}\sin\theta$$

$$s = \frac{2v^2}{g}\cos\theta\sin\theta = \frac{v^2}{16}\sin\theta\cos\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$$

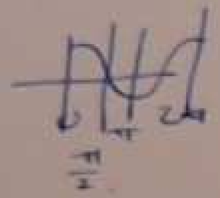
distance travelled $\boxed{s = \frac{v^2}{32}\sin 2\theta}$

a) estimate Δs if $\Delta\theta = 3^\circ$.

$$\Delta s \approx \frac{ds}{d\theta}\Delta\theta$$

$$\frac{ds}{d\theta} = \frac{v^2}{32}\cos(2\theta)\cdot 2$$

$$= \frac{v^2}{32}\cos(2\theta)$$



$v = 110$

$\theta = 24$

$$\approx \frac{110^2}{32}\cos(48)\cdot 3$$

b) estimate Δs if $\Delta v = 2$

$$\Delta s \approx \frac{ds}{dv}\Delta v = \frac{2v}{32}\sin 2\theta\,\Delta v$$

$$= \frac{2\cdot 110}{32}\sin(48)\cdot 2$$

$$s = \frac{2v^2}{g}\cos\theta\sin\theta = \frac{v^2}{16}\sin\theta\cos\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$

distance travelled $\boxed{s = \frac{v^2}{32}\sin 2\theta}$

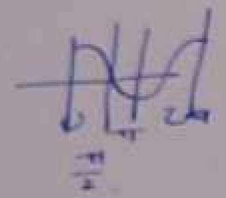
a) estimate Δs if

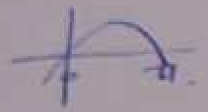
$\Delta\theta = 3^\circ$ ← working in radians

$$\Delta s \approx \frac{ds}{d\theta}\underbrace{\Delta\theta}_{\text{radians}}$$

$$\frac{ds}{d\theta} = \frac{v^2}{32}\cos(2\theta)\cdot 2$$

$$= \frac{v^2}{32}\cos(2\theta)$$





$v = 110$

$\theta = 24$

$$\approx \frac{110^2}{32}\cos(\overset{\frac{2\times 24\times\pi}{180}}{\cancel{48}})\cdot\cancel{3}\cdot\frac{3\times\pi}{180}$$

b) estimate Δs if $\Delta v = 2$

$$\Delta s \approx \frac{ds}{dv}\Delta v = \frac{2v}{32}\sin 2\theta\,\Delta v$$

$$= \frac{2\cdot 110}{32}\sin(\overset{\frac{2\times 24\times\pi}{180}}{\cancel{48}})\cdot 2$$

Unpin Video Recording...

WW 4.2 Q6 ~~$f(x) = x^3 + 4x^2 - 7x - 3$~~ $f(\theta) = \sqrt{12}\,\theta - \sqrt{6}\sec\theta$ (5)

$[0, \frac{\pi}{3}]$

find abs max, min

$\left(\frac{1}{\cos\theta}\right)' = \left((\cos\theta)^{-1}\right)'$

find critical points, solve $f'(\theta) = 0$.

$-(\cos\theta)^{-2} \cdot -\sin\theta$

$\frac{\sin\theta}{\cos^2\theta}$.

$f'(\theta) = \sqrt{12} - \sqrt{6}\sec\theta\tan\theta = 0$.

$\boxed{\sqrt{6}}$ $\sqrt{6}\,\frac{\sin\theta}{\cos^2\theta} = \sqrt{12}$.

$\frac{\sin\theta}{\cos^2\theta} = \sqrt{2}$

$\sin\theta = \sqrt{2} - \sqrt{2}\sin^2\theta$

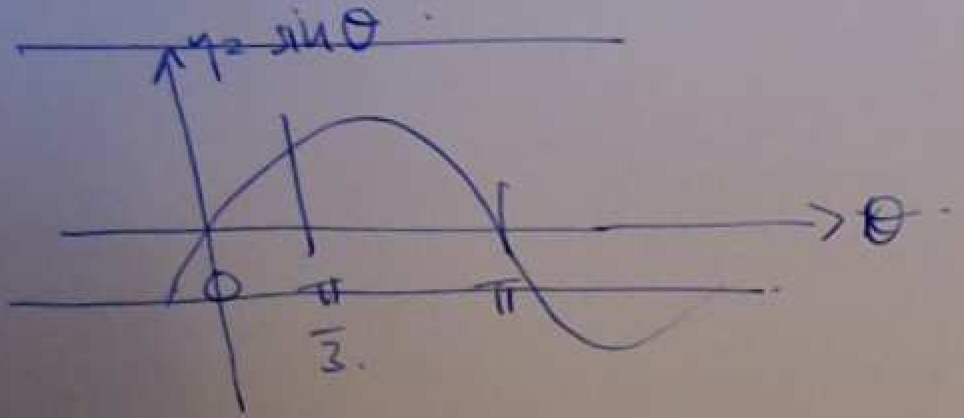
$\sqrt{2}\sin^2\theta - \sin\theta - \sqrt{2} = 0$.

$\sin\theta = \sqrt{2}\,\underbrace{\cos^2\theta}_{1-\sin^2\theta}$.

$\sin\theta = \frac{1 \pm \sqrt{1+8}}{2\sqrt{2}}$.

Joseph Maher

$$\sin\theta = \frac{1 \pm \sqrt{9}}{2\sqrt{2}} = \frac{1 \pm 3}{2\sqrt{2}} = \frac{-2}{2\sqrt{2}}, \frac{4}{2\sqrt{2}} = \frac{-1}{\sqrt{2}}, \frac{2}{\sqrt{2}}.$$



no solutions so
no critical points.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$\sqrt{1}/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{4}/2$
$\cos\theta$					

$f(0) = -\sqrt{6}\sec(0) = -\sqrt{6}$. abs. min

$f(\frac{\pi}{3}) = \sqrt{12} \cdot \frac{\pi}{3} - \sqrt{6}\sec(\frac{\pi}{3})$.

$= \sqrt{12} \cdot \frac{\pi}{3} - \sqrt{6} \cdot \sqrt{2} = \sqrt{12}(\frac{\pi}{3} - 1)$. abs max

4

$$s = \frac{2v^2}{g}\cos\theta\sin\theta = \frac{v^2}{16}\sin\theta\cos\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$

distance travelled $\boxed{s = \frac{v^2}{32}\sin 2\theta}$

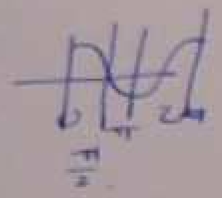
a) estimate Δs if $\Delta\theta = 3^\circ$

working in radians

$$\Delta s \approx \frac{ds}{d\theta}\Delta\theta \quad \text{radians}$$

$$\frac{ds}{d\theta} = \frac{v^2}{32}\cos(2\theta)\cdot 2$$

$$= \frac{v^2}{32}\cos(2\theta)2$$

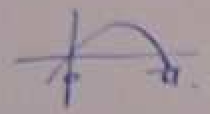


$v = 110$

$\theta = 24$

$\frac{2\times 24\times\pi}{180}$

$$\approx \frac{110^2}{32}\cos(\cancel{48})\cdot\cancel{3}\cdot\frac{3\times\pi}{180}\times 2$$



b) estimate Δs if $\Delta v = 2$

$$\Delta s \approx \frac{ds}{dv}\Delta V = \frac{2v}{32}\sin 2\theta\,\Delta V$$

$\frac{2\times 24\times\pi}{180}$

$$= \frac{2.110}{32}\sin(\cancel{48})\cdot 2$$

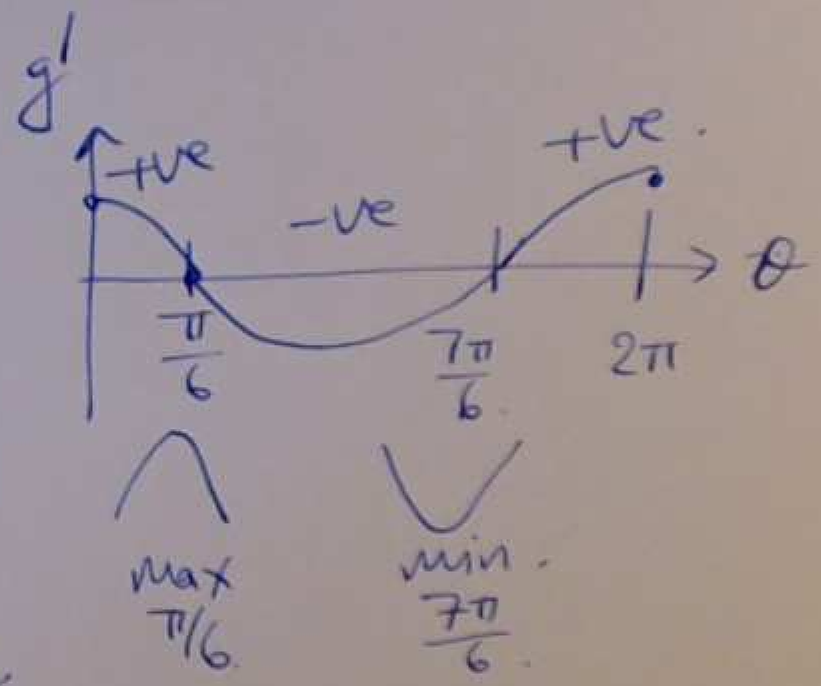
$g(\theta) = 5\sin\theta + 5\sqrt{3}\cos\theta$

$g'(\theta) = 5\cos\theta - 5\sqrt{3}\sin\theta$

critical points $\frac{\pi}{6}, \frac{7\pi}{6}$.

$g'(\cancel{\theta}\,0) = 5 - 0 = 5 > 0$

$g'(\pi) = -5 < 0$



g

$g'(\theta) > 0$ $[0, \frac{\pi}{6})$ increasing

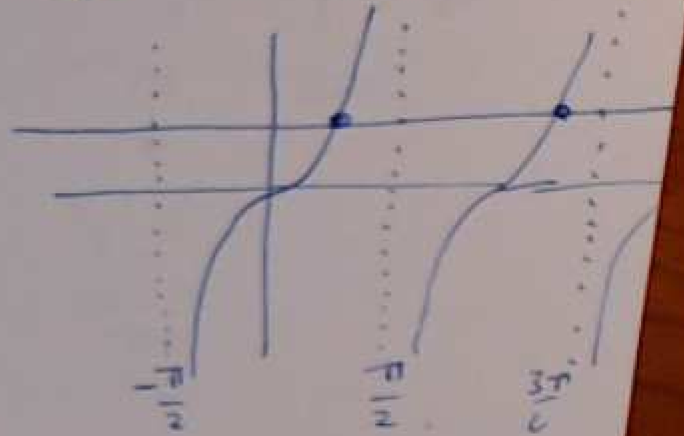
$g'(\theta) < 0$ $(\frac{\pi}{6}, \frac{7\pi}{6})$ decreasing

$g'(\theta) > 0$ $(\frac{7\pi}{6}, 2\pi)$ increasing

HW 4.3 Q7

7

$$g(\theta) = 5\sin\theta + \underset{\substack{25\times 3 \\ 5\sqrt{3}}}{\sqrt{75}}\cos\theta \qquad [0, 2\pi]$$



$$g'(\theta) = 5\cos\theta + 5\sqrt{3} \cdot -\sin\theta$$

find critical points: solve $g'(\theta) = 0$

$$\cancel{5}\cos\theta - \cancel{5}\sqrt{3}\sin\theta = 0$$

$$\cancel{5}\cos\theta = \sqrt{3}\sin\theta$$

$$\frac{1}{\sqrt{3}} = \tan\theta \qquad \theta = \frac{\pi}{6} \pm n\pi \qquad n \in \mathbb{Z}$$

$$\theta = \frac{\pi}{6}, \frac{\pi}{6} + \pi = \frac{\pi}{6}, \frac{7\pi}{6}$$

9

WW 4.3 Q8

$$y = x - \ln(x^7) = x - 7\ln(x)$$

recall

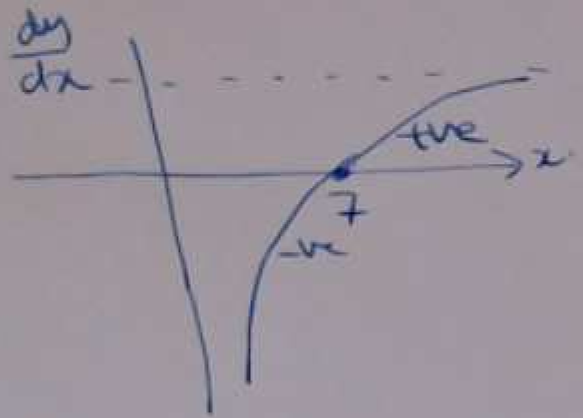
$\ln(a^b) = b\ln(a)$;

$$\frac{dy}{dx} = 1 - 7/x.$$

critical point: solve $\frac{dy}{dx} = 0$

$$1 - \frac{7}{x} = 0.$$

$$1 = \frac{7}{x} \qquad x = 7.$$

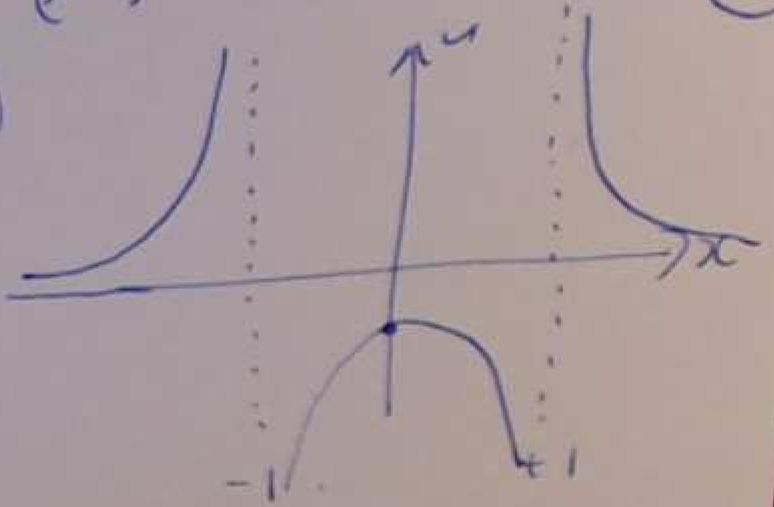


$\frac{dy}{dx} < 0 \leftrightarrow y$ ~~increasing~~ decreasing $(0, 7)$

$\frac{dy}{dx} > 0 \leftrightarrow y$ increasing $(7, \infty)$.

$x = 7$ is a local max.

Example $f(x) = \dfrac{1}{x^2-1} = \dfrac{1}{(x-1)(x+1)} = (x^2-1)^{-1}$



① vertical asymptotes at $x = \pm 1$

② $f'(x) = -(x^2-1)^{-2} \cdot (2x)$

$= \dfrac{-2x}{(x^2-1)^2}$

critical points: $f'(x) = 0$

$\dfrac{-2x}{(x^2-1)^2} = 0 \qquad x = 0$

③ $f'(x) < 0 \quad x > 0 \quad f$ decreasing

$f'(x) > 0 \quad x < 0 \quad f$ increasing

④ $f''(x) = \dfrac{(x^2-1)^2 \cdot (-2) - (-2x) \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4}$

$= \dfrac{(x^2-1)(-2) + (8x^2)}{(x^2-1)^3}$

$= \dfrac{6x^2+2}{(x^2-1)^3}$

inflection points: $6x^2+2=0$

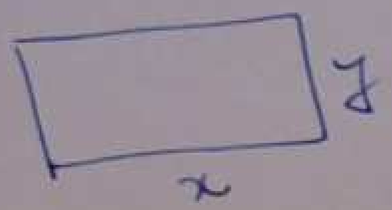
$\boxed{\text{no}}$ $\qquad x^2 = -3$

no solution

⑤ horizontal asymptotes $\lim_{x \to \pm\infty} f(x) = 0$

§4.6 Optimization

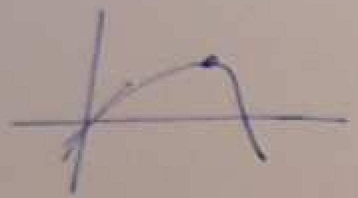
Example a piece of wire of length L is bent into a rectangle. What is the largest possible area?



$$A = xy$$

$$L = 2x + 2y \leftarrow \frac{L}{2} - x = y$$

$$A = xy = x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2$$

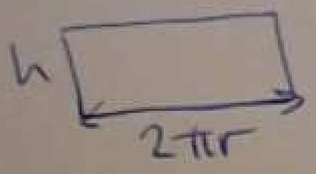


critical points $\frac{dA}{dx} = \frac{L}{2} - 2x = 0 \qquad x = \frac{L}{4},\ y = \frac{L}{4}$

max area $A = \frac{L}{4} \times \frac{L}{4} = \frac{L^2}{16}$ (max area is when rectangle is a square)

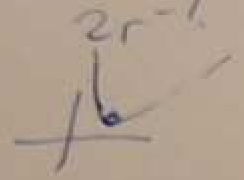
Example what shape of cylindrical can minimizes the surface area? total volume $V = 1 ft^3$.

$$\begin{cases} V = \text{height} \times \text{area of base} = h\pi r^2 \ (*) \\ A = 2\pi r^2 + 2\pi r h. \end{cases}$$



$(*)$ $V = h\pi r^2$ $\qquad h = \frac{V=1}{\pi r^2}$.

$$A = 2\pi r^2 + \cancel{2\pi r}\cdot\frac{1}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$$

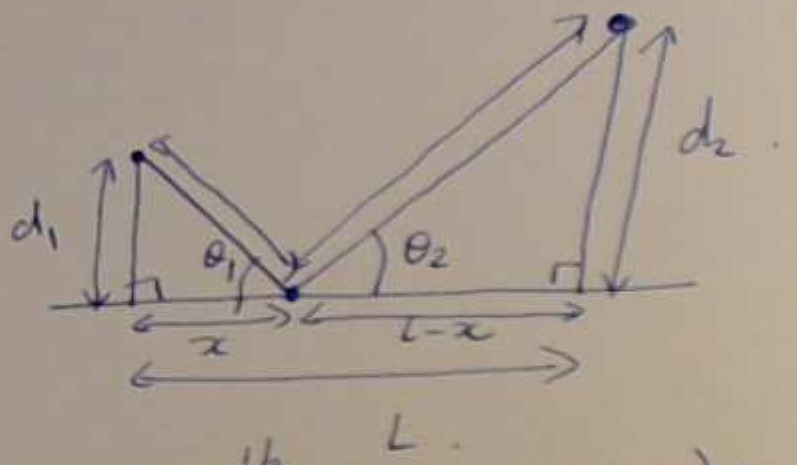


find critical points $\frac{dA}{dr} = 4\pi r + -2r^{-2} = 0$

$4\pi r = \frac{2}{r^2}$ $\qquad r^3 = \frac{2}{4\pi}$ $\qquad r = \sqrt[3]{\frac{1}{2\pi}}$ min.

.... work out h, A.

Example: a ball bounces off a wall show that the path which minimizes distance has equal angles of incidence and reflection.



$f(x)$ = distance travelled

$$f(x) = \sqrt{d_1^2+x^2} + \sqrt{d_2^2+(L-x)^2}.$$

$$f'(x) = \cancel{\tfrac{1}{2}}(d_1^2+x^2)^{-1/2}(\cancel{2}x) + \cancel{\tfrac{1}{2}}\left(d_2^2+(L-x)^2\right)^{-1/2}\cdot \cancel{2}(L-x).(-1) = 0$$

$f'(x)=0$: $\Leftrightarrow$ $$\frac{x}{\sqrt{d_1^2+x^2}} = \frac{L-x}{\sqrt{d_2^2+(L-x)^2}}.$$

$$\cos\theta_1 = \cos\theta_2. \Rightarrow \theta_1 = \theta_2.$$