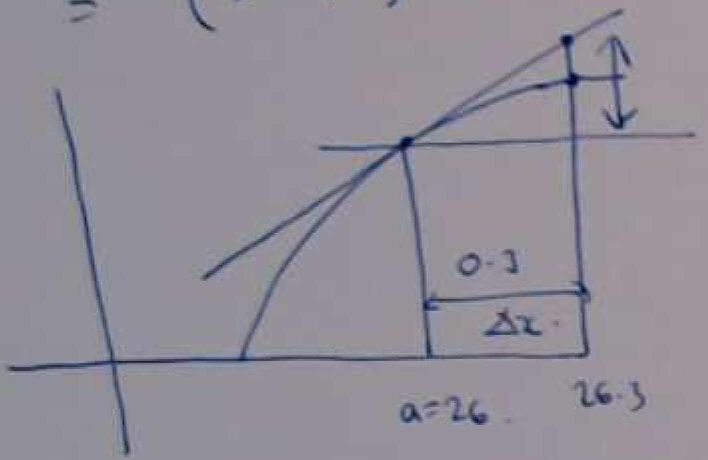
WW 4.1 Q1 . (1)

$f(x) = \sqrt{x-10}$ $\quad a = 26 \quad \Delta x = 0.3.$

$= (x-10)^{1/2}.$



$f(26) = \sqrt{26-10} = \sqrt{16} = 4.$

Q: $f(26.3)$?

$f(x) \simeq f(26) + f'(26)\Delta x.$

$\uparrow$

$f(x) \approx 4 + \underbrace{\frac{1}{8} 0.3}$

$\Delta f = \frac{1}{8} 0.3 = 0.024.$

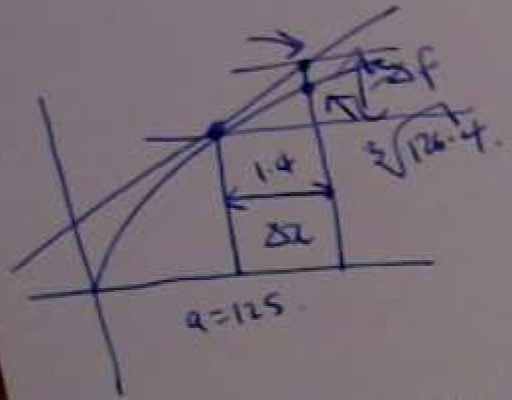
$f'(x) = \frac{1}{2}(x-10)^{-1/2}(x-10)'.$

$= \frac{1}{2\sqrt{x-10}}$

$f'(26) = \frac{1}{2\sqrt{26-10}} = \frac{1}{2\sqrt{16}} = \frac{1}{8}.$

WW 4.1 Q2

$f(x) = \sqrt[3]{x} = x^{1/3}$ $\quad a = 125 \quad \Delta x = 1.4$



$f(125) = 5$

$f(126.4) \approx f(125) + f'(125)\Delta x$

$\quad\quad\quad\quad\quad 5 \quad + \quad\quad\quad 1.4$

$\approx 5 + \boxed{\frac{1.4}{75}}$

$f'(x) = \frac{1}{3} x^{-2/3}$

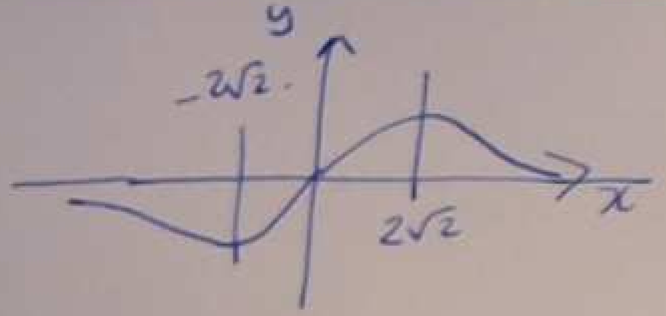
$= \frac{1}{3\sqrt[3]{x^2}}$

$\Delta f \approx \frac{1.4}{75}$

$f'(125) = \frac{1}{3 \cdot 5^2} = \frac{1}{75}$

WW 4.2 Q2

$$f(x) = \frac{x}{x^2+8}$$



$$f'(x) = \frac{(x^2+8)(x)' - x(x^2+8)'}{(x^2+8)^2}$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$= \frac{x^2+8-x(2x)}{(x^2+8)^2} = \frac{8-x^2}{(x^2+8)^2} = 0$$

$x^2 = 8 \quad x = \pm 2\sqrt{2}$

positive critical point is $2\sqrt{2}$

WW 4.2 Q1

$$f(x) = x^3 + 4x^2 - 7x - 3.$$

find critical points: solve $f'(x) = 0$

$$f'(x) = 3x^2 + 8x - 7$$

$3 \times 4 \times 7$

$$x = \frac{-8 \pm \sqrt{64 + 84}}{16} = -\frac{1}{2} \pm \frac{\sqrt{148}}{16}$$

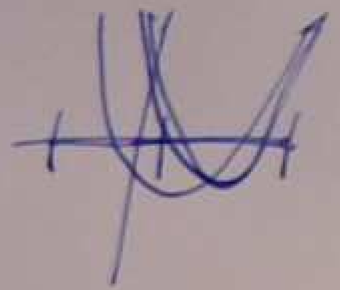
$$= -\frac{1}{2} \pm \frac{\sqrt{37}}{8}$$

$$-\frac{1}{2} + \frac{\sqrt{37}}{8}, \quad -\frac{1}{2} - \frac{\sqrt{37}}{8}$$

WW 4.2 Q4

$$f(x) = 5x^2 - 20x + 10$$

$$f'(x) = 10x - 20$$



critical points: solve $f'(x) = 0$

$$10x - 20 = 0$$

$$x = 2.$$

$$f(2) = 5 \times 4 - 40 + 10$$

$$= -10.$$

find extreme values on

$[0, 4]$

$f(0) = 10$

$f(2) = -10$

$f(4) = 10.$

abs max $= 10$

abs min $= -10$.

$[-3, 1]$

$f(-3) = 45 + 60 + 10 = 115.$ ← abs max.

$f(1) = -5.$ ← abs min.

~~tttt~~ WW 4.2 Q1. 6

$f(x) = x^3 + 7x^2 - 3x$

$f'(x) = 3x^2 + 14x - 3$.

$$\frac{-14 \pm \sqrt{14^2 + 36}}{6}.$$

$$\frac{-14 + \sqrt{132}}{6}, \quad \frac{-14 - \sqrt{132}}{6}.$$

$$\begin{array}{r} 14 \\ 14 \\ \hline 16 \\ 40 \\ 140 \\ \hline 196 \\ 36 \\ \hline 132 \end{array}$$

WW 4.3 Q8

$y = x - \ln(x^7)$

$f(x) = x - 7\ln(x)$.

$f'(x) = 1 - \frac{7}{x}$.

critical point: $f'(x) = 0$

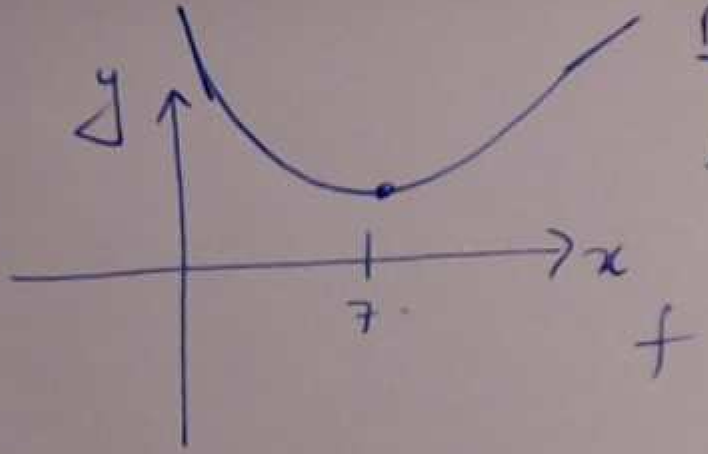
$1 - \frac{7}{x} = 0$

$x = 7$.

local min.

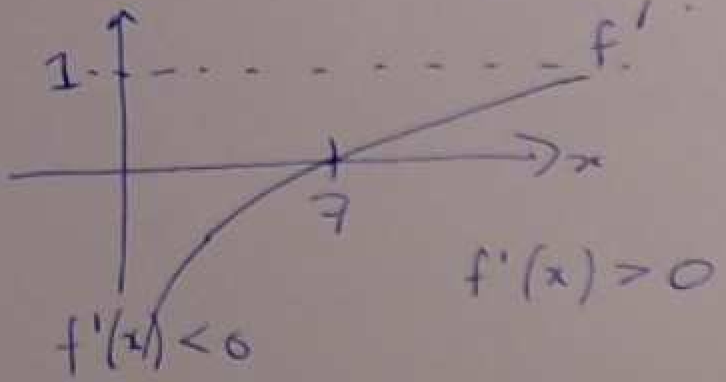
increasing $(7, \infty)$.

decreasing $(0, 7)$.



recall

$\ln(a^b) = b\ln(a)$



HW 4.4 Q4

$f(x) = (x^2-7)e^x$ $\quad (fg)' = f'g + fg'$

$f'(x) = (x^2-7)'e^x + (x^2-7)(e^x)'$.

$f'(x) = 2x e^x + (x^2-7)e^x = (x^2+2x-7)e^x$.

$f''(x) = (x^2+2x-7)'e^x + (x^2+2x-7)(e^x)'$.

$= (2x+2)e^x + (x^2+2x-7)e^x$.

$= (x^2+4x-5)e^x = (x+5)(x-1)e^x = 0$

$f''(x) = 0$ when $x = -5, +1 \leftarrow$ pts of inflection.

$(-\infty, -5)$ $f''(x) > 0$ concave up.

$(-5, 1)$ $f''(x) < 0$ concave down.

$(1, \infty)$ $f''(x) > 0$ concave up.

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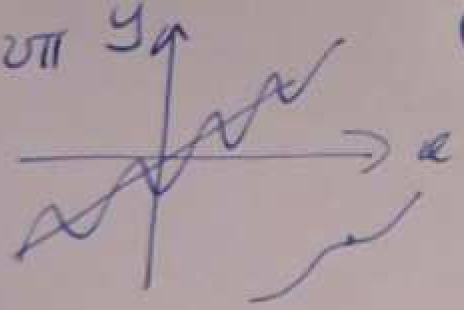
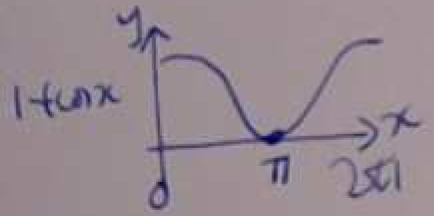
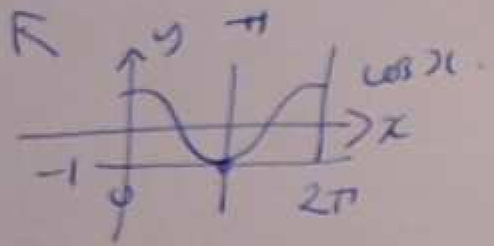
WW 4.4 Q8 $f(x) = x+\sin(x)$ $0 \le x \le 2\pi$



$f'(x) = 1+\cos(x)$

$f''(x) = -\sin(x)$





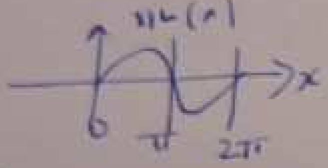
critical points: $f'(x)=0$ $1+\cos(x)=0$

$\cos(x)=-1$

one critical point at $x=\pi$.

second derivative test: $f''(\pi) = -\sin(\pi) = 0$

no information.



$[0,\pi)$ $f'(x)>0$ increasing

$(\pi,2\pi]$ $f'(x)>0$ increasing

inflection points: $f''(x) = -\sin(x) = 0$ $x=0,\pi,2\pi$.

$(0,\pi)$ $f''(x)>0$ concave up

$(\pi,2\pi)$ $f''(x)<0$ concave down.

§4.5 l'Hôpital's rule

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$ if $f(x), g(x)$ differentiable and this is indeterminate form $\frac{0}{0}, \frac{\infty}{\infty}$

~~not~~ $\frac{1}{0}$.

then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$ ← warning: this is not differentiation $\frac{f}{g}$

why does this work?

$\frac{0}{0}$.

$f(x) \simeq f'(a)(x-a)$

$$\frac{f(x)}{g(x)} \simeq \frac{f'(a)\cancel{(x-a)}}{g'(x)\cancel{(x-a)}} \simeq \frac{f'(a)}{g'(a)}.$$

$g(x) \simeq g'(a)(x-a)$

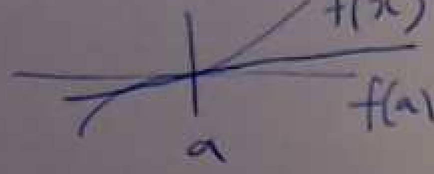
§4.5 l'Hôpital's rule

$\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ if $f(x), g(x)$ differentiable
and this is indeterminate form $\dfrac{0}{0}, \dfrac{\infty}{\infty}$ ~~not~~ $\dfrac{1}{0}$.

then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ ← warning: this is not differentiating $\dfrac{f}{g}$

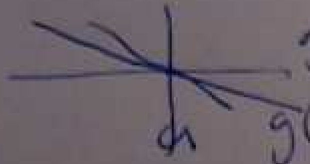
why does this work?

$\dfrac{0}{0}$



$f(x) \simeq f'(a)(x-a)$

$\dfrac{f(x)}{g(x)} \simeq \dfrac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}}$



$g(x) \simeq g'(a)(x-a)$ $\qquad \simeq \dfrac{f'(a)}{g'(a)}$

§4.5 l'Hôpital's rule

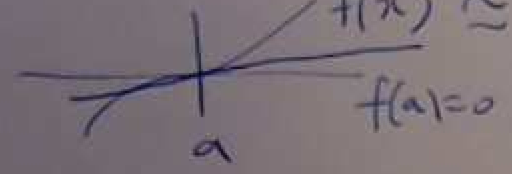
$$\lim_{x\to a} \frac{f(x)}{g(x)}$$ if $f(x), g(x)$ differentiable and this is indeterminate form $\frac{0}{0}, \frac{\infty}{\infty}$ — not $\frac{1}{0}$.

then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$
← warning: this is not differentiating $\frac{f}{g}$

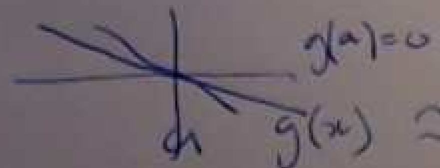
why does this work?

$\frac{0}{0}$

$f(x) \simeq f'(a)(x-a)$



$$\frac{f(x)}{g(x)} \simeq \frac{f'(a)\cancel{(x-a)}}{g'(x)\cancel{(x-a)}}$$



$g(x) \simeq g'(a)(x-a)$

$$\simeq \frac{f'(a)}{g'(a)}$$

examples

① $$\lim_{x\to 0+} x\ln(x) = \lim_{x\to 0}\frac{\ln(x)}{1/x}$$ indeterminate form $\frac{\infty}{\infty}$.

x^{-1}.

l'H: $$\lim_{x\to 0+}\frac{1/x}{-x^{-2}} = \lim_{x\to 0+}\frac{1/x}{-1/x^{2}} = \lim_{x\to 0+} -\frac{x}{1} = 0.$$

② $$\lim_{x\to 0}\frac{e^{x}-x-1}{\cos x-1} \underset{l'H}{=} \lim_{x\to 0}\frac{e^{x}-1}{-\sin x}$$ indeterminate form $\frac{0}{0}$.

$$\underset{l'H}{=} \lim_{x\to 0}\frac{e^{x}}{-\cos x} = \frac{1}{-1} = -1.$$

(3) $$\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} \qquad \frac{0}{0}.$$

$$\underset{l'H}{=} \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x}. \qquad \frac{0}{0} \text{ indeterminate form.}$$

$$\underset{l'H}{=} \lim_{x\to 0} \frac{+\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{1+1-0} = 0.$$

(4) $$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{x\ln(x)}. \qquad (x)^x = (e^{\ln(x)})^x$$

$\exp(x)$ cts $$= e^{\boxed{\lim_{x\to 0+} x\ln x.}} \leftarrow 0 = e^0 = 1.$$

examples

(1) $\lim_{x\to 0+} x$

l'H: $\lim_{x\to 0+}$

(2) $\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1}$

$\underset{l'H}{=} \lim_{x\to 0} \frac{e}{-\cos x}$

comparing growth rates of functions

Q: which grows faster $(\ln(x))^2$ or $\sqrt{x}$?

$$\lim_{x\to\infty} \frac{\sqrt{x}}{(\ln(x))^2} \qquad \frac{\infty}{\infty} \text{ indeterminate form.}$$

$$\underset{l'H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{2(\ln(x))\frac{1}{x}} = \lim_{x\to\infty} \frac{x^{1/2}}{4\ln(x)} \qquad \frac{\infty}{\infty}$$

$$\underset{l'H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{4/x} = \lim_{x\to\infty} \frac{1}{8}\sqrt{x} = \infty.$$

conclusion is $\sqrt{x}$ grows faster.

Thm e^x grows faster than any polynomial. (14)

Proof

$$\lim_{x\to\infty} \frac{x^n}{e^x} \underset{l'H}{=} \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} \quad \frac{\infty}{\infty}$$

$\frac{\infty}{\infty}$

$$\underset{l'H}{=} \lim_{n\to\infty} \frac{n(n-1)x^{n-2}}{e^x} \underset{l'H}{=} \cdots\cdots = \lim_{n\to\infty} \frac{n!}{e^x} = 0.$$

so e^x grows faster than x^n for any n. $\square$

§4.6 Graph sketching and asymptotes

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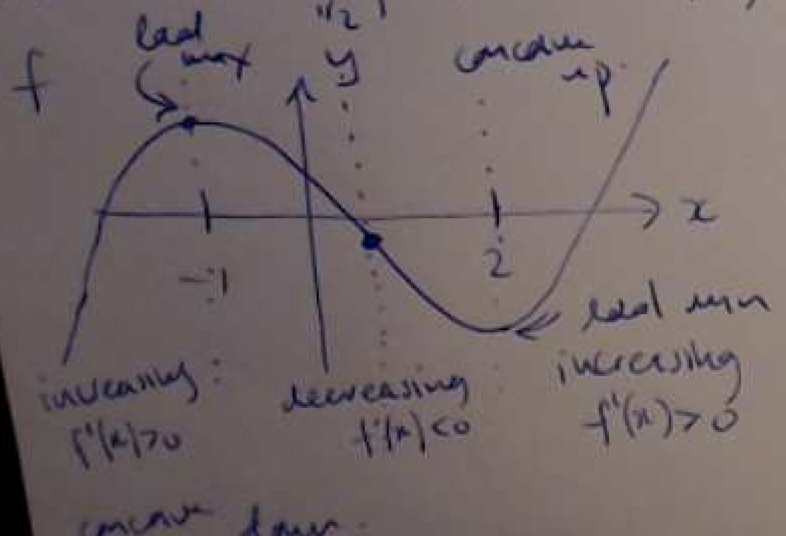
Example sketch graph of $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3 = f(x)$

helpful info:
- critical points $f'(x) = 0$
- increasing/decreasing $f'(x) \leftarrow$ sign
- concave up/down $\leftarrow$ sign $f''(x)$.

find critical points: $f'(x) = x^2 - x - 2 = (x-2)(x+1)$.



$f''(x) = 2x - 1$

$f''(x) = 0$: $x = \frac{1}{2}$.

Example $f(x) = (4x - x^2)e^x$. (16)

$$f'(x) = (4x - x^2)'e^x + (4x - x^2)(e^x)'$$

$$= (4 - 2x)e^x + (4x - x^2)e^x$$

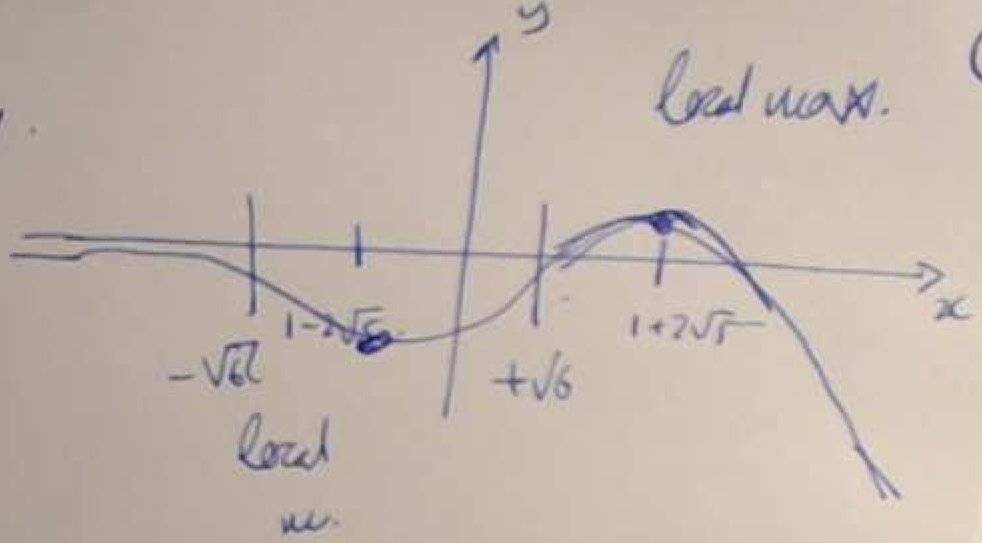
$$= (4 + 2x - x^2)e^x.$$

critical points: solve $f'(x) = 0$

$-x^2 + 2x + 4$ $x = \dfrac{-2 \pm \sqrt{4+16}}{-2}$

$x = 1 \pm \sqrt{20}$

$x = 1 \pm 2\sqrt{5}$



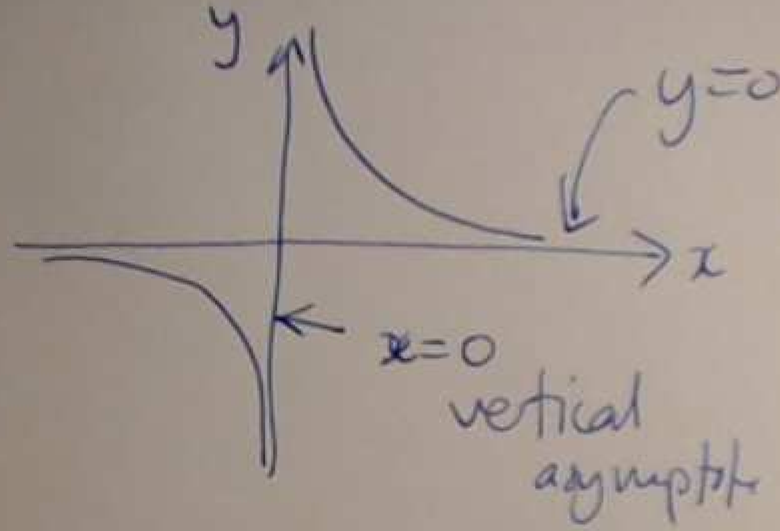
$$f''(x) = (4 + 2x - x^2)'e^x + (4 + 2x - x^2)(e^x)'$$

$$= (2 - 2x)e^x + (4 + 2x - x^2)e^x$$

$$= (6 - x^2)e^x$$ inflection points $x = \pm\sqrt{6}$

Asymptotes

Example $y = \frac{1}{x}$



Def^{n} $x=c$ is a vertical asymptote if $\lim_{x \to c^{\pm}} f(x) = \pm\infty$.

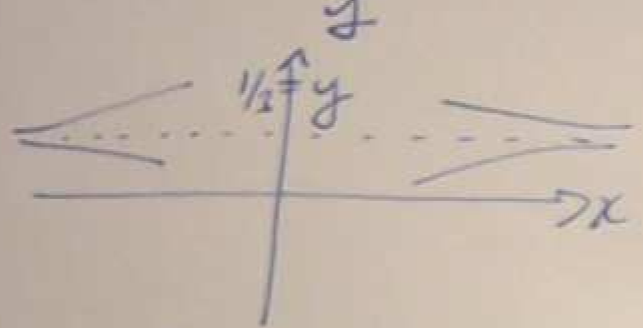
$y=c$ is a horizontal asymptote if $\lim_{x \to \pm\infty} f(x) = c$.

observation rational function $\frac{p(x)}{q(x)}$ have

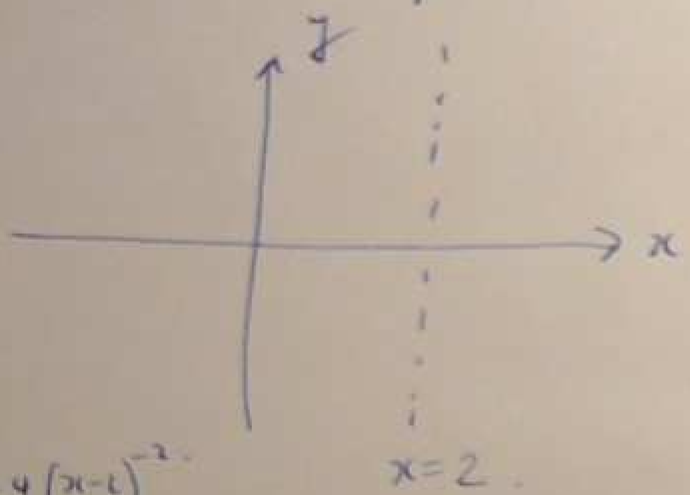
horizontal asymptotes if $\deg p \leq \deg q$

Example $f(x) = \dfrac{x^2+x+1}{3x^2+2} \sim \dfrac{x^2}{3x^2} \sim \dfrac{1}{3}.$

$$\lim_{x\to\infty} f(x) = \frac{1}{3}$$



Example $f(x) = \dfrac{3x+2}{2x-4}.$



① find vertical asymptotes.

$\Leftrightarrow$ denominator $= 0$ $\quad x = 2.$

② $f'(x) = \dfrac{(2x-4)(3x+2)' - (3x+2)(2x-4)'}{(2x-4)^2}$

$-4(x-2)^{-2}$

"

$f'(x) = \dfrac{(2x-4)(3) - (3x+2)2}{(2x-4)^2} = \dfrac{-16}{(2x-4)^2} = \dfrac{-4}{(x-2)^2}$ ← always negative.

$f(x)$ decreasing

③ $f''(x) = 8(x-2)^{-3} = \dfrac{8}{(x-2)^3}$

$x>2 \quad f''(x)>0$ concave up.

$x<2 \quad f''(x)<0$ concave down

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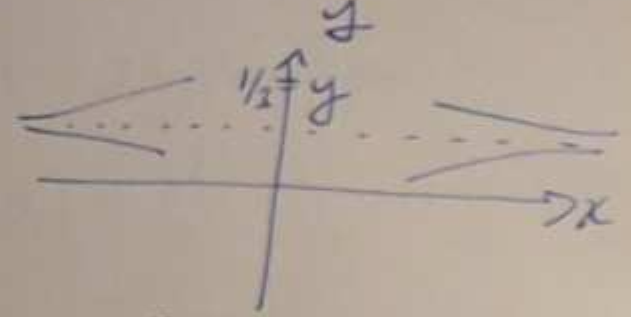
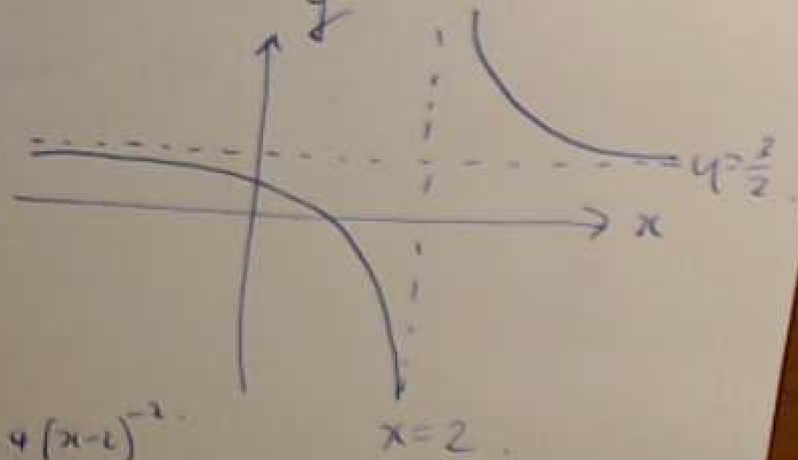
Example $f(x) = \dfrac{x^2+x+1}{3x^2+2} \sim \dfrac{x^2}{3x^2} \sim \dfrac{1}{3}$. (18)

$\lim_{x\to\infty} f(x) = \dfrac{1}{3}$



Example $f(x) = \dfrac{3x+2}{2x-4}$.



① find vertical asymptotes.

$\Leftrightarrow$ denominator $= 0$ $\quad x = 2$.

② $f'(x) = \dfrac{(2x-4)(3x+2)' - (3x+2)(2x-4)'}{(2x-4)^2}$

$-4(x-2)^{-2}$

$f'(x) = \dfrac{(2x-4)(3) - (3x+2)2}{(2x-4)^2} = \dfrac{-16}{(2x-4)^2} = \dfrac{-4}{(x-2)^2}$ ← always negative.

f(x) decreasing

③ $f''(x) = 8(x-2)^{-3} = \dfrac{8}{(x-2)^3}$ $\quad x>2 \quad f''(x)>0$ concave up

$x<2 \quad f''(x)<0$ concave down

example

Asy

Example

Def^{n} $x=$

$y=c$ is a hor

observation

horizontal asymp

Joseph Maher

4 horizontal asymptotes

$$f(x) = \frac{3x+2}{2x-4} \sim \frac{3}{2}.$$

horizontal asymptotes

$$\lim_{x \to +\infty} f(x) = \frac{3}{2}$$

$$\lim_{x \to -\infty} f(x) = \frac{3}{2}.$$

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④ horizontal asymptotes (19)

$$f(x) = \frac{3x+2}{2x-4} \sim \frac{3}{2}.$$

horizontal asymptotes

$$\lim_{x \to +\infty} f(x) = \frac{3}{2}$$

$$\lim_{x \to -\infty} f(x) = \frac{3}{2}.$$

Joseph Maher