WW 3.8 Q1

$\frac{d}{dx}(x^3y^4)$. $x^3(y(x))^4$.

product rule: $(fg)' = f'g + fg'$.

chain rule: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$.

$$(x^3y^4)' = (x^3)'y^4 + x^3(y^4)'$$

$$3x^2y^4 + x^3(y(x))^4$$

$$= 3x^2y^4 + x^2\,4y^3\underbrace{\frac{dy}{dx}}_{D}.$$

$$4(\underbrace{\quad}_{g(x)})^3 \cdot y'(x)$$

WW 3.8 Q7

$$y = \cos^{-1}(5x^6)$$

1 chain rule

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}.$$

$$\cos(y) = 5x^6.$$ ← implicit diff.

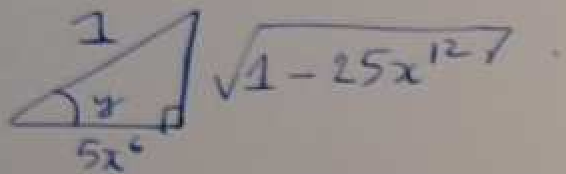
$$\cos(y(x)) = 5x^6.$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

$$-\sin(y(x)) \cdot \frac{dy}{dx} = 30x^5$$

$$\frac{dy}{dx} = -\frac{30x^5}{\sin(y)}$$

$$\cos(y) = 5x^6.$$

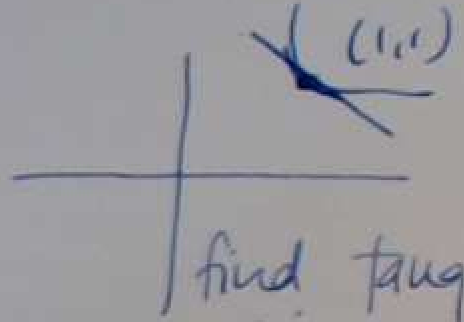


$$\sin(y) = \sqrt{1-25x^{12}}.$$

$$\frac{dy}{dx} = -\frac{30x^5}{\sqrt{1-25x^{12}}}$$

WW 3.8 Q9

$$x^{2/3} + y^{2/4} = 2 \qquad (1,1)$$



find tangent line

find $\frac{dy}{dx}$

$$x^{2/3} + \left(y(x)\right)^{1/2} = 2$$

$$\frac{2}{3} x^{-1/3} + \frac{1}{2}\left(y(x)\right)^{-1/2} \frac{dy}{dx} = 0$$

1 1

$$\frac{2}{3} + \frac{1}{2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4}{3}$$

$$y - y_0 = m(x - x_0)$$

$$y - 1 = -\frac{4}{3}(x - 1)$$

$$y = -\frac{4}{3}(x - 1) + 1$$

WW 3.10 Q2

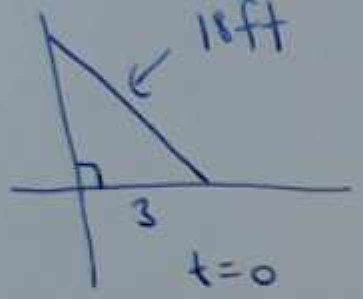
$\frac{dr}{dt} = 16 \text{ in/min}$ $\qquad V = \frac{4}{3}\pi r^3$

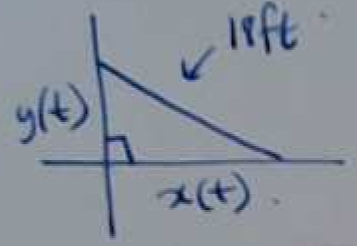
find $\frac{dV}{dt}$ when $r = 20$

implicit diff $\longrightarrow V(t) = \frac{4}{3}\pi(r(t))^3$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}(\underbrace{r(t)}_{20})^2 \cdot \underbrace{\frac{dr}{dt}}_{16} = 4\pi \cdot (20)^2 \cdot 16$$

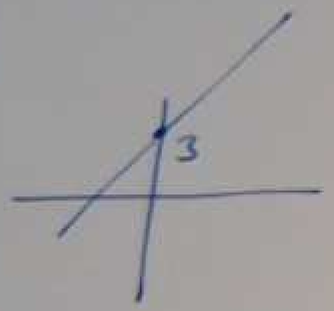
WW 3.10 Q4





$$\boxed{\begin{aligned} x(0) &= 3 \\ \frac{dx}{dt} &= 3 \end{aligned}}$$

$x(t)$



$$\boxed{x^2 + y^2 = 18^2}$$

$$(x(t))^2 + (y(t))^2 = 18^2$$

$$\boxed{2\underbrace{(x(t))}_{6}\underbrace{\frac{dx}{dt}}_{3} + 2\underbrace{(y(t))}_{12\sqrt{2}}\frac{dy}{dt} = 0}$$

$t=1$

$$\frac{dy}{dt} = \frac{-\cancel{2}\times\cancel{6}\times 3}{2\ \cancel{2}\times\cancel{12}\sqrt{2}} = -\frac{3}{2\sqrt{2}}$$

$$x(t) = 3+3t$$

$$x(1) = 6$$

$$x^2+y^2=18^2$$

$$6^2+y^2=18^2$$

$$y^2 = \sqrt{18^2-6^2}$$

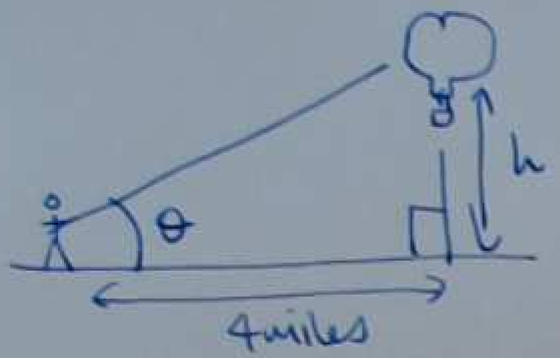
$$\sqrt{(6\times 3)^2-6^2}$$

$$y^2 = \sqrt{6^2}\sqrt{3^2-1}$$

$$= 6\sqrt{8}$$

$$= 12\sqrt{2}$$

HW 3.10 Q6



$$\theta = \frac{\pi}{4} \qquad \frac{d\theta}{dt} = 0.1 \text{ rad/min}$$

$$\frac{h}{4} = \tan\theta$$

$$\frac{h(t)}{4} = \tan(\theta(t)) \leftarrow \text{diff}$$

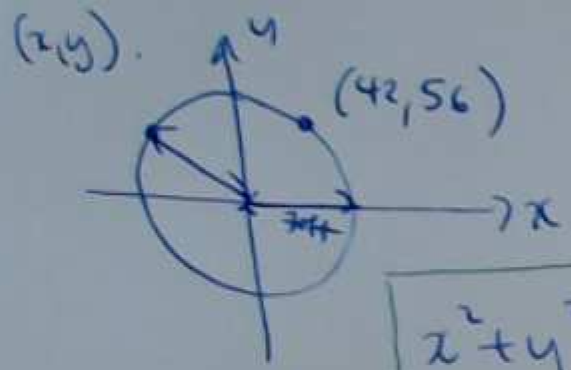
$$\frac{1}{4}\frac{dh}{dt} = \sec^2(\underbrace{\theta(t)}_{\pi/4})\,\underbrace{\frac{d\theta}{dt}}_{0.1}$$

$$\sec^2\left(\frac{\pi}{4}\right) = \frac{1}{\cos^2\left(\frac{\pi}{4}\right)} = \frac{1}{\left(1/\sqrt{2}\right)^2} = 2$$

$$\frac{1}{4}\frac{dh}{dt} = 2 \cdot 0.1$$

$$\frac{dh}{dt} = 0.8 \text{ miles/min}$$

WW. 3.10 Q8



$$\frac{dx}{dt} = 17 \text{ ft/s}$$

$$\boxed{x^2+y^2 = 70^2}$$

diff $\rightarrow (x(t))^2 + (y(t))^2 = 70^2$

$$\cancel{2}\underbrace{x(t)}_{42}\underbrace{\frac{dx}{dt}}_{17} + \cancel{2}\underbrace{y(t)}_{56}\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{42 \cdot 17}{56} \text{ ft/s}$$

recall **Thm** First derivative test

If $f(x)$ is differentiable, and $f'(c) = 0$

then if $f'(x)$ changes from $+$ve to $-$ve at c

$\Rightarrow$ local max.

$-$ve to $+$ve at c

$\Rightarrow$ local min

Example classify the critical points of

$f(x) = \cos^2 x + \sin x$ on $[0, \pi]$

find the critical points.

$f(x) = \cos^2 x + \sin x$ $[0, \pi]$.

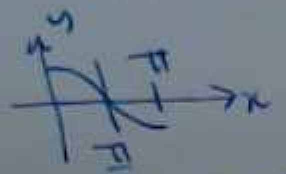
$f'(x) = 2\cos x \cdot -\sin x + \cos x$.

solve $f'(x) = 0$: $-2\sin x \cos x + \cos x = 0$.

$\cos(x)(1 - 2\sin x) = 0$.

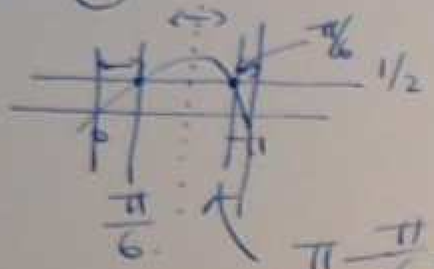
$\cos(x) = 0$ or $1 - 2\sin x = 0$

$\sin x = \frac{1}{2}$



$x = \frac{\pi}{2}$.

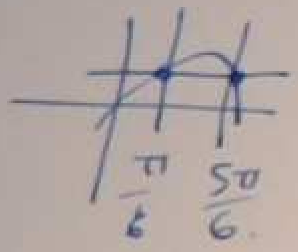
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
		$\frac{1}{2}$			



$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

critical points $\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

$$f'(x) = \cos(x) \cdot (1 - 2\sin x).$$



~~sin x~~ cos(x)	+	+	−	−
$1-2\sin x$	+	−	−	+

$0 \quad \pi/6 \quad \pi/2 \quad \frac{5\pi}{6} \quad \pi$

$f'(x)$	+	−	+	−

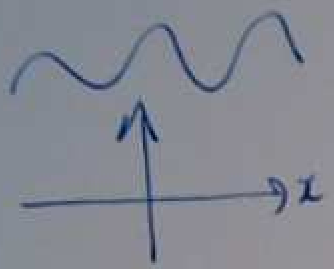
local max — local min — local max.

§4.4 Second derivative test

__recall__

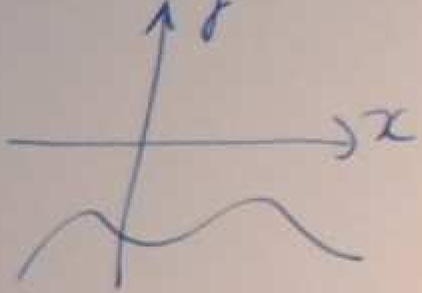
$f(x) > 0$

f positive



$f(x) < 0$

f negative

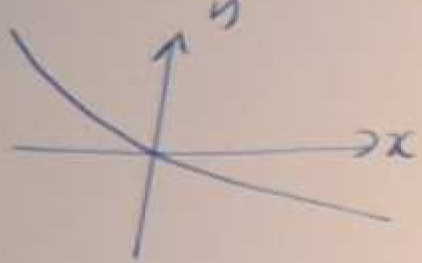


$f'(x) > 0$

f increasing

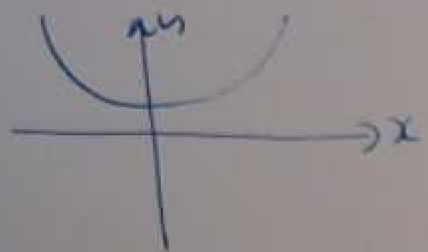
$f'(x) < 0$

f decreasing



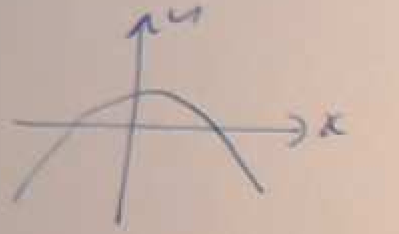
$f''(x) > 0$

concave up



$f''(x) < 0$

concave down



__mnemonic__

concave:  down  up

Q how do slopes change?

concave up.

concave down

$\Leftrightarrow$ slopes are increasing

$\Leftrightarrow$ $f''(x) > 0$

$\Leftrightarrow$ slopes are decreasing

$\Leftrightarrow$ $f''(x) < 0$

Defn Let $f(x)$ be differentiable on (a, b)

then $f(x)$ is concave up $\Leftrightarrow$ $f''(x) > 0$

$f(x)$ is concave down $\Leftrightarrow$ $f''(x) < 0$.

Defn A point of inflection is where the graph changes from concave up to concave down (or vice versa)

Note: point of inflection $\Rightarrow$ $f''(x) = 0$

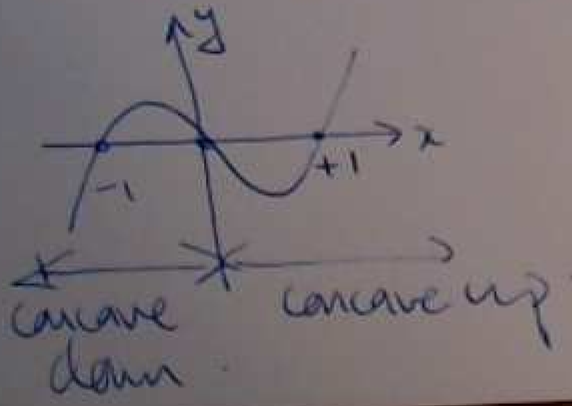
$$\not\Leftarrow$$

Example $f(x) = x^3 - x = x(x^2-1) = x(x-1)(x+1)$

$f'(x) = 3x^2 - 1$

$f''(x) = 6x$ $\quad$ $f''(x) > 0 \quad x > 0$

$f''(x) < 0 \quad x < 0$



Def^n^: A point of inflection is where the graph changes from concave up to concave down (or vice versa)

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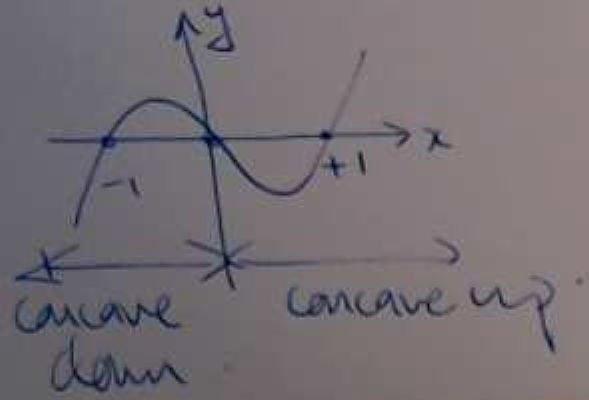
$\nLeftarrow$

Example $f(x) = x^3 - x = x(x^2-1) = x(x-1)(x+1)$

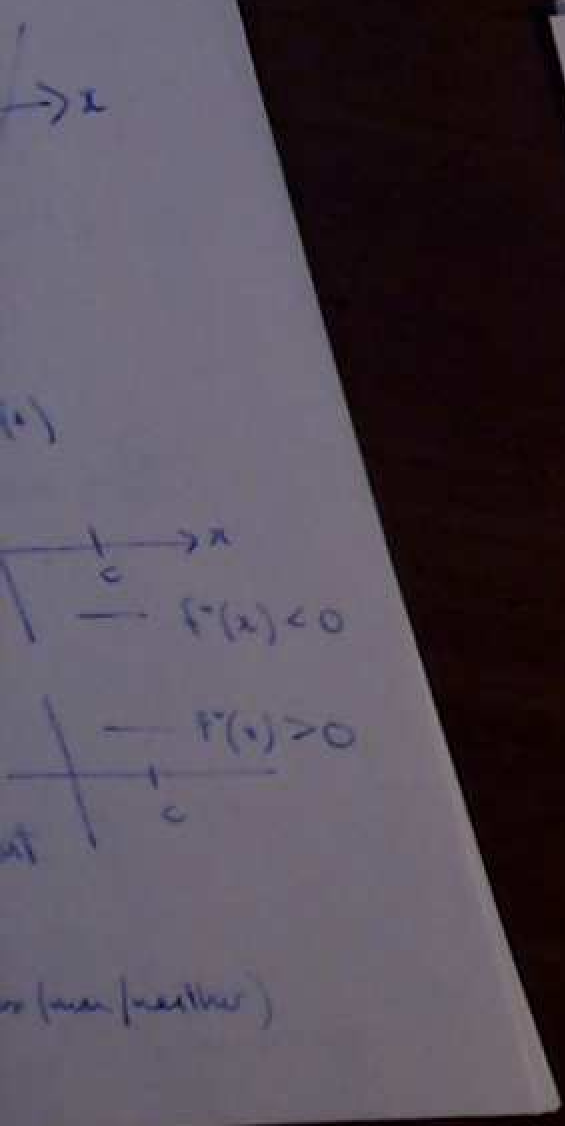


$f'(x) = 3x^2 - 1$

$f''(x) = 6x$

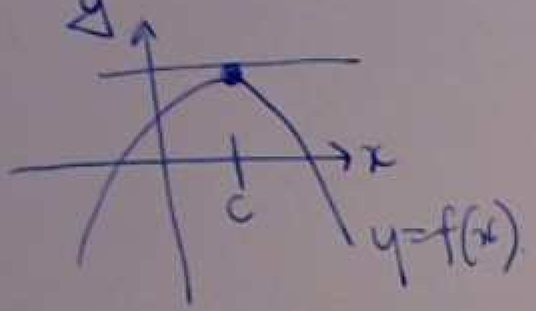
$f''(x) > 0 \quad x > 0$

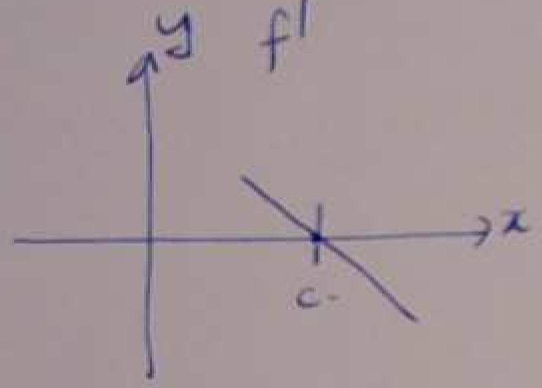
$f''(x) < 0 \quad x < 0$

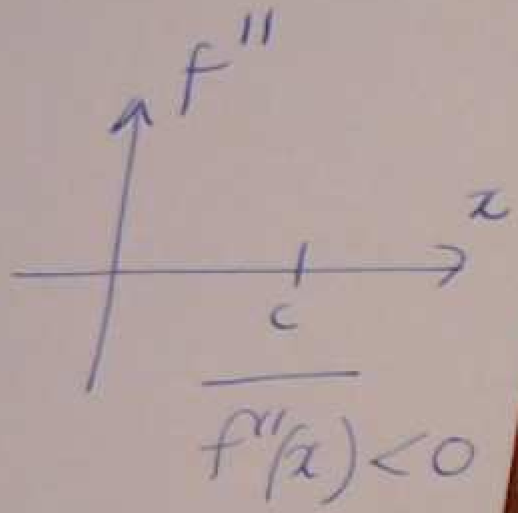


Second derivative test

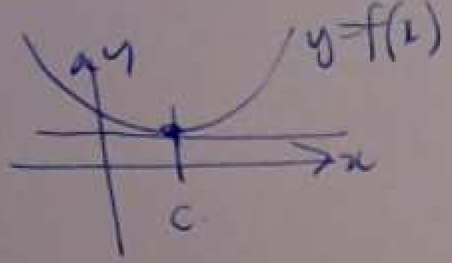
local max:

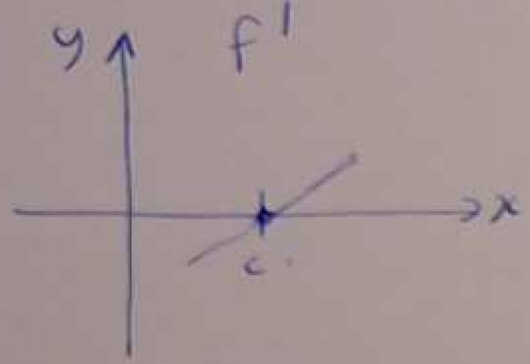


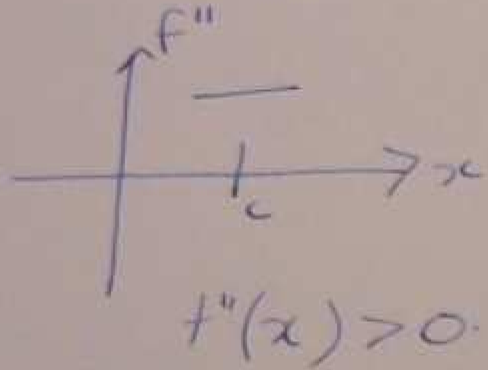




local min

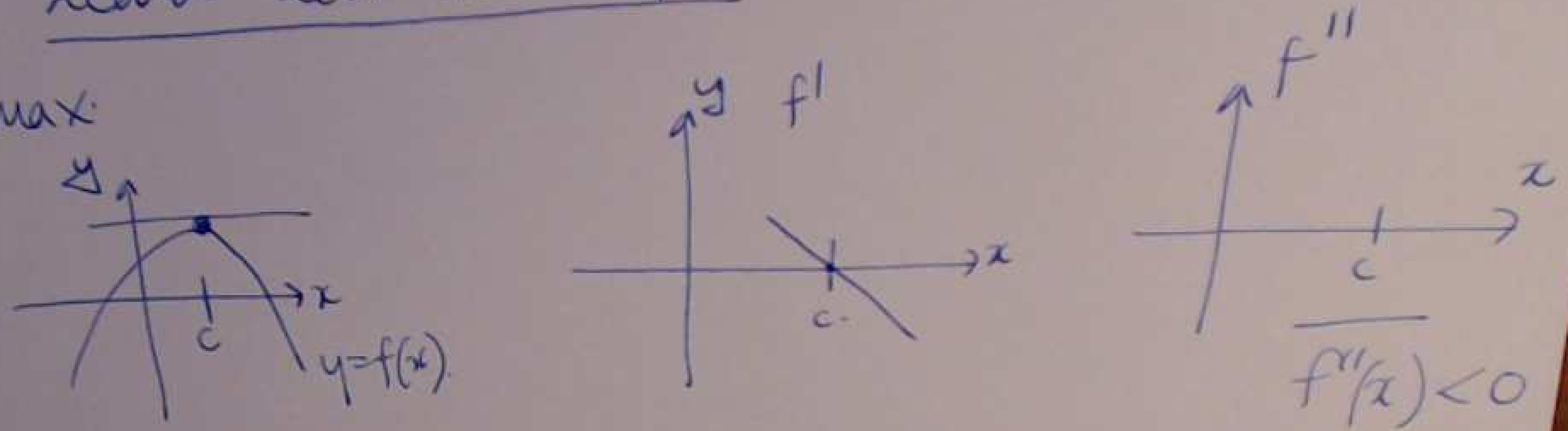




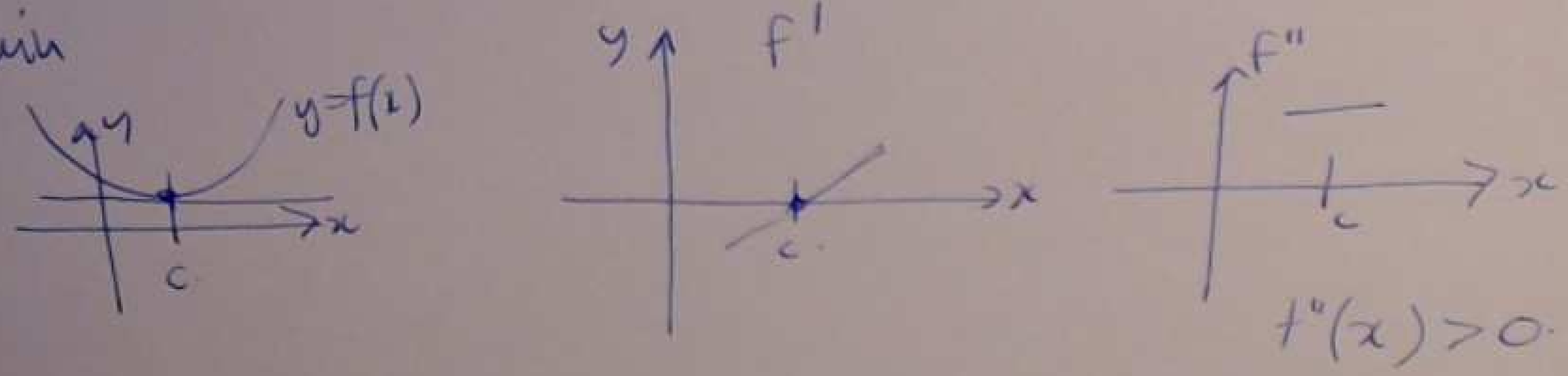


~~the~~ Second derivative test

local max:



local min



Thm suppose $f(x)$ is differentiable and c is a critical point

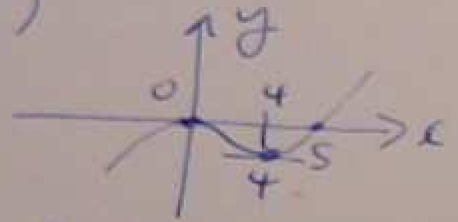
If $f''(c) > 0 \Rightarrow$ local min

$f''(c) < 0 \Rightarrow$ local max.

$f''(c) = 0$ no information! (could be a local max/min/neither).

Example $f(x) = x^5 - 5x^4 = x^4(x-5)$

$f'(x) = 5x^4 - 20x^3$



find critical points: solve $f'(x) = 0$ $\quad 5x^4 - 20x^3 = 0$

$5x^3(x-4) = 0$

$x = 0, \ x = 4$

local max.

$f(a)=g(a)=0$
(or $f(a)=g(a)=\pm\infty$).

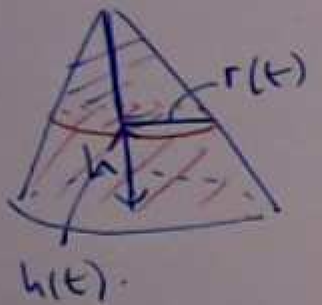
WW 3.10 Q3.

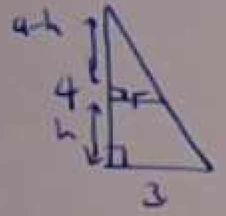
$1\,m^3/min$

$\frac{dV}{dt}=1.$ $V=\frac{1}{3}h\pi r^2.$

$t=0$ $V=0$

$V=\frac{1}{3}\times 4\times\pi\times 3^2 = 12\pi.$





$\frac{4}{3}=\frac{4-h}{r}$

$$\boxed{r=\frac{3(4-h)}{4}}$$
$=3-\frac{3}{4}h$

missing volume is

$V=\frac{1}{3}(4-h)\pi r^2$

$\frac{1}{3}(4-h)\pi\left(\frac{3}{4}\right)^2(4-h)^2$

$\frac{\pi\,3}{16}(4-h)^3$

$$\boxed{V = 12\pi - \frac{\pi\,3}{16}(4-h)^3}$$

$$V = 12\pi - \frac{3\pi}{16}(4-h)^3.$$

$$\frac{dV}{dt} = -\frac{3\pi}{16}\,3(4-h)^2 \cdot -1 = \frac{9\pi}{16}(4-h)^2.$$

$h = 1$ $\quad \frac{81\pi}{16}.$

$h = 2.$ $\quad 9\pi.$

18

$$V = 12\pi - \frac{3\pi}{16}(4-h)^3. \qquad V(t) = 12\pi - \frac{3\pi}{16}(4-h(t))^3.$$

$$\underbrace{\frac{dV}{dt}}_{1} = + \frac{3\pi}{16}\, 3(4-h)^2 \cdot \frac{dh}{dt} = \frac{9\pi}{16}(4-h)^2$$

~~$h=1$ $\frac{81\pi}{16}$.~~

~~$h=2$. 9π.~~

$$1 = \frac{9\pi}{16}(4-h)^2 \frac{dh}{dt}.$$

$$h = 1 \qquad 1 = \frac{81\pi}{16}\frac{dh}{dt} \qquad \frac{dh}{dt} = \frac{16}{81\pi}$$

$$h = 2 \qquad 1 = \frac{9\pi \cdot 4}{16}\frac{dh}{dt} \qquad \frac{dh}{dt} = \frac{4}{9\pi}.$$

§4.5. l'Hôpital's rule

Thm suppose $f(x)$ and $g(x)$ are differentiable

and $f(a) = g(a) = 0$ (or $f(a) = g(a) = \pm\infty$).

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

warning ① this is not the quotient rule.

② $\frac{f(a)}{g(a)}$ must be indeterminate form $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$.

can't use this on $\lim_{x \to 0} \frac{1}{x}$

Examples

① $\displaystyle \lim_{x\to 1} \frac{x^3-1}{x-1} \overset{\text{l'Hopital}}{=} \lim_{x\to 1} \frac{3x^2}{1} = 3.$

② $\displaystyle \lim_{x\to 1} \frac{x^{100}-1}{x-1} = \lim_{x\to 1} \frac{100x^{99}}{1} = 100.$

③ $\displaystyle \lim_{x\to \frac{\pi}{2}} \frac{\cos^2 x}{\sin x - 1} \overset{\text{l'H}}{=} \lim_{x\to \frac{\pi}{2}} \frac{2\cos x \cdot -\sin x}{\cos x}$

$\displaystyle = \lim_{x\to \frac{\pi}{2}} -2\sin x = -2.$