recall rules for differentiation.

$$(kf)' = kf'$$

$$(f+g)' = f' + g'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

f	f'
x^n	nx^{n-1}
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2 x$
$\vdots$	$\vdots$

chain rule.

$$(f(g(x)))' = f'(g(x)) \cdot \overset{g}{\cancel{f}}{}'(x).$$

example. $f(x) = e^{\sqrt{x}}$ $\quad g(x) = \sqrt{x} = x^{1/2} \quad g'(x) = \frac{1}{2}x^{-1/2}$

$f(x) = e^x \quad f'(x) = e^x.$

$$f'(x) = e^{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2}.$$

Alternative notation

$$f(g(x)) \longleftrightarrow f(u), \quad u = g(x).$$

$$\frac{df}{dx} = f'(g(x))\cdot g'(x) \longleftrightarrow f'(u)\frac{du}{dx}.$$

$$\longleftrightarrow \frac{df}{du}\cdot\frac{du}{dx}.$$

$$\boxed{\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}}$$

mnemonic:
"cancel fractions"

Examples. $\frac{d}{dx}(\sin(x)) = \cos(x)$ $\leftarrow$ radians.

$$\left(\sin\left(\frac{\pi x}{180}\right)\right)' = \cos\left(\frac{\pi x}{180}\right)\frac{\pi}{180}.$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

$f(x) = \sin(x)$ $\quad$ $f'(x) = \cos(x)$

$g(x) = \frac{\pi x}{180}$ $\quad$ $g'(x) = \frac{\pi}{180}$

Examples $\frac{d}{dx}(\sin(x)) = \cos(x) \leftarrow$ radians.

$$\left(\sin\left(\frac{\pi x}{180}\right)\right)' = \cos\left(\frac{\pi x}{180}\right)\frac{\pi}{180}.$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

$f(x) = \sin(x) \qquad f'(x) = \cos(x)$

$g(x) = \frac{\pi x}{180} \qquad g'(x) = \frac{\pi}{180}$

<u>Examples</u>

(1) $\frac{d}{dx}\left(\sin^2(x)\right) = 2\sin(x)\cos(x)$.

" $(\sin(x))^2$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$f(x) = x^2 \qquad f'(x) = 2x$

$g(x) = \sin(x) \qquad g'(x) = \cos(x)$

(2) $\frac{d}{dx}\left(-\cos^2(x)\right) = -\frac{d}{dx}\left(\cos^2(x)\right) = \underbrace{-2\cos(x)\cdot(-\sin(x))}_{2\sin(x)\cos(x)}$

$(f(g(x)))' = f'(g(x)) \cdot g'(x) = 2$

$f(x) = x^2 \qquad f'(x) = 2x$

$g(x) = +\cos(x) \qquad g'(x) = -\sin(x)$

$\cancel{(-\cos(x))^2 = \cos^2 x}$.

$\sin^2 x + \cos^2 x = 1$. $\sin^2 x$, $-\cos^3 x$

$\sin^2 x = 1 - \cos^2 x$ ← same up to constant

$$(f(g(x)))' = f'(g(x)) \cdot g'(x).$$

<u>Proof</u> (of chain rule).

$$[f(g(x))]' = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}.$$

write this as: $$\lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \underbrace{\boxed{\frac{g(x+h) - g(x)}{h}}}_{g'(x)}.$$

$$\lim_{h\to 0} \frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} \; \underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}_{g'(x)}$$

set $k = g(x+h)-g(x)$.

<u>key fact</u>: if g differentiable $\Rightarrow$ g is continuous.

so ~~means~~ implies $k \to 0$ as $h \to 0$.

$$\underbrace{\lim_{k\to 0} \frac{f(g(x)+k)-f(g(x))}{k}} \cdot g'(x).$$

$$= f'(g(x)) \cdot g'(x) \qquad \square$$

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(6)

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \quad \underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}_{g'(x)}.$$

set $k = g(x+h) - g(x)$.

<u>key fact</u>: if g differentiable $\Rightarrow$ g is continuous.

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$$\underbrace{\lim_{\cancel{h}\, k\to 0} \frac{f(g(x)+k) - f(g(x))}{k}} \cdot g'(x).$$

$$= f'(g(x)) \cdot g'(x) \quad \square$$

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Examples

$$\frac{d}{dx}\left((g(x))^n \right) = n\,(g(x))^{n-1} \cdot g'(x)$$

$$\frac{d}{dx}\left(e^{g(x)} \right) = e^{g(x)} \cdot g'(x)$$

$$\frac{d}{dx}\left(f(ax+b) \right) = f'(ax+b) \cdot (ax+b)'$$

$$= f'(ax+b)\,a.$$

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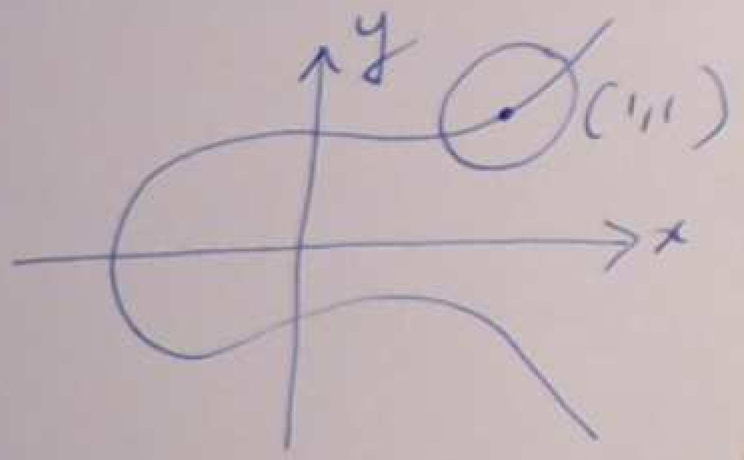
§ 3.8 Implicit differentiation

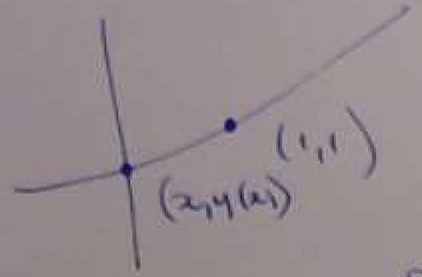
consider : $y^4 + xy = x^3 - x + 2$

→ equation. can't solve for x, y explicitly easily.



this does define a function locally.

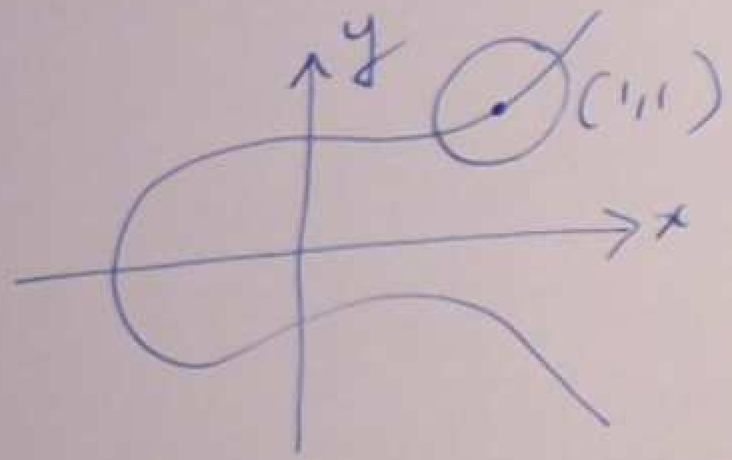
locally:



$(x, y(x))$.

could do it other way round $(x(y), y)$.

§ 3.8 Implicit differentiation



consider: $y^4 + xy = x^3 - x + 2$

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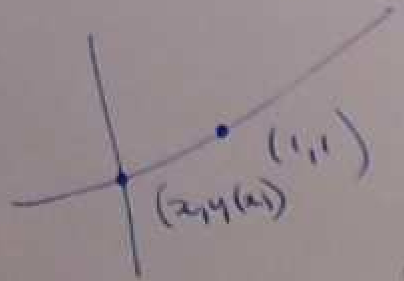
locally:



$(x, y(x))$.

could do it other way round $(x(y), y)$.

$$y^4 + xy = x^3 - x + 2$$

$(x, y(x))$

$$(y(x))^4 + x\,y(x) = x^3 - x + 2$$

↑ differentiate wrt x using chain rule on $y(x)$
WRT

$$4(y(x))^3\, y'(x) + (x)'\, y(x) + x\cdot(y(x))' = 3x^2 - 1$$

$$4(y(x))^3\, y'(x) + y(x) + x\,y'(x) = 3x^2 - 1$$

$$y'(x)\left[4y(x)^3 + x\right] + y = 3x^2 - 1$$

$$\boxed{y'(x) = \frac{3x^2 - 1 - y}{4y^3 + x}}$$

$$y'(1) = \frac{3-1-1}{4+1} = \frac{1}{5}$$

$$y^4 + xy = x^3 - x + 2$$

$(x, y(x))$.

$$(y(x))^4 + x\,y(x) = x^3 - x + 2.$$

↑ differentiate wrt x using chain rule on $y(x)$.

WRT

$$4(y(x))^3\,y'(x) + (x)'\,y(x) + x\cdot(y(x))' = 3x^2 - 1.$$

$$4(y(x))^3\,y'(x) + y(x) + x\,y'(x) = 3x^2 - 1.$$

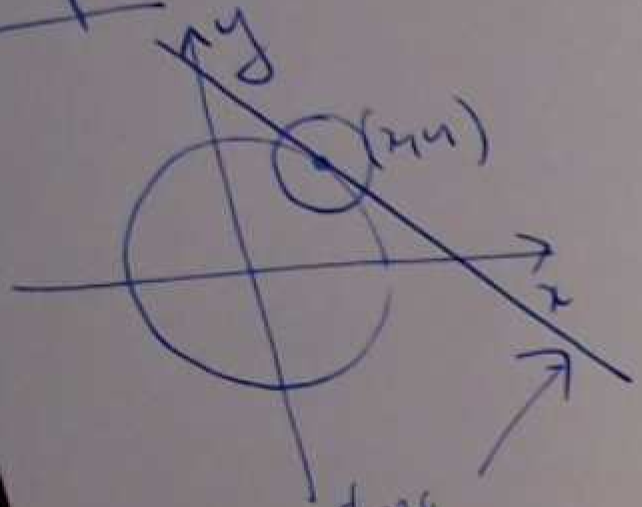
$$y'(x)\left[4y(x)^3 + x\right] + y = 3x^2 - 1.$$

$$\boxed{y'(x) = \frac{3x^2 - 1 - y}{4y^3 + x}}$$

$$y'(1) = \frac{3-1-1}{4+1} = \frac{1}{5}.$$

example

$$x^2 + y^2 = 1.$$



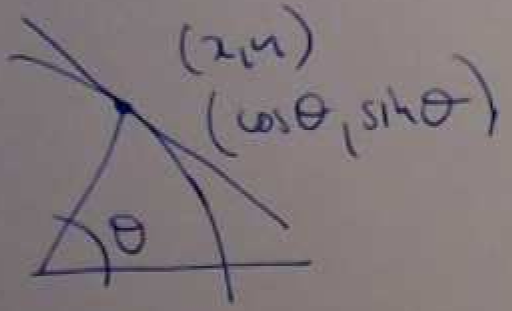
slope is $y'(x)$

$$(x, y(x)).$$

$$x^2 + (y(x))^2 = 1. \leftarrow \text{differentiate wrt } x.$$

$$2x + 2y(x) \cdot y'(x) = 0.$$

$$y'(x) = -\frac{x}{y}$$



$$y'(\cos\theta) = -\frac{\cos\theta}{\sin\theta}$$

$$= -\cot\theta.$$

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Application: derivatives of inverse functions

Example $y = \ln(x)$

$e^{\ln(x)} = x$

$\ln(e^x) = x$

$e^{y(x)} = x$. $\leftarrow$ implicit diff wrt x.

$e^{y(x)} \cdot y'(x) = 1$

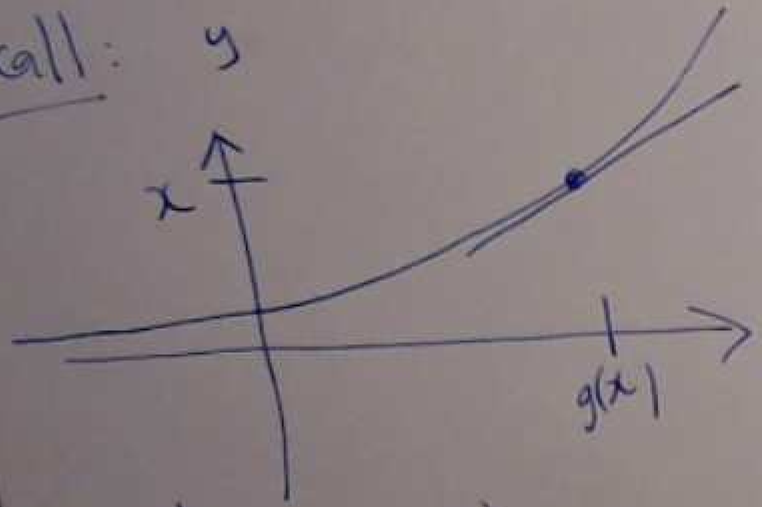
$y'(x) = \frac{1}{e^{y(x)}} = \frac{1}{x}$.

$$\boxed{\textbf{Thm} \quad \frac{d}{dx}(\ln(x)) = \frac{1}{x}}$$

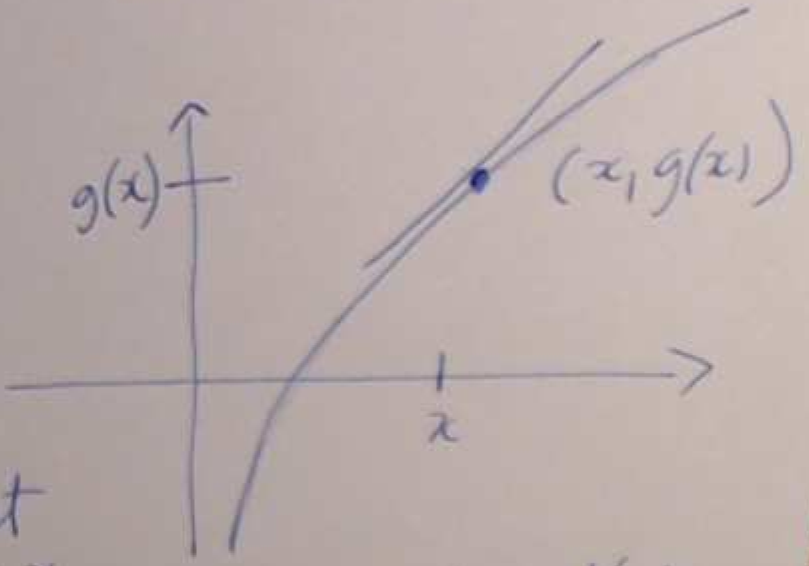
<u>Thm</u> suppose $f(x)$ is differentiable, with inverse $f^{-1}(x) = g(x)$, then $g'(x) = \dfrac{1}{f'(g(x))}$

as long as $f'(g(x)) \neq 0$.

<u>recall</u>:



$\longleftrightarrow$ reflect in $y = x$.



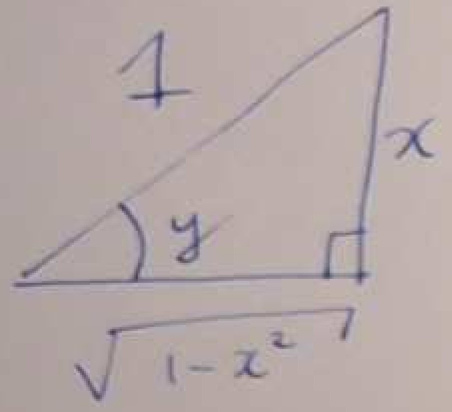
slope at $g'(x)$.

$\dfrac{1}{\text{slope}}$ at $g(x)$ $\longleftarrow$

$= \dfrac{1}{f'(g(x))}$ $\square$.

Thm $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$

Proof (of $\sin^{-1}(x)$).



$y = \sin^{-1}(x)$

$\sin(y(x)) = x \leftarrow$ implicit diff

$\cos(y(x)) \cdot y'(x) = 1$

$y'(x) = \frac{1}{\cos(y(x))} = \frac{1}{\sqrt{1-x^2}}$. $\square$

Thm $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}.$

Proof $y = \tan^{-1}(x).$

$\tan(y) = x. \leftarrow$ implicit diff wrt x.

$\frac{\sin^2 y + \cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$

$\tan^2 y + 1 = \sec^2 y$

$\sec^2(y) \cdot y' = 1$

$y' = \frac{1}{\sec^2 y}.$

$y' = \frac{1}{\tan^2 y + 1} = \frac{1}{1+x^2}$ $\square$.

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Thm $\frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{1+x^2}$.

$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx}\left(\csc^{-1}(x)\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

Example. $\frac{d}{dx}\left(\tan^{-1}\left(e^{2x}\right)\right)$

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(16)

$\frac{d}{dx}\left(\tan^{-1}(e^{2x})\right) \leftarrow$ need to use chain rule twice.

$(f(g(x)))' = f'(g(x))\,g'(x)$.

$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$.

$\frac{1}{1+(e^{2x})^2} \cdot \left(e^{2x}\right)'$.

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(2x) = 2$.

$\frac{1}{1+e^{4x}} \cdot e^{2x} \cdot (2x)'$

$= \frac{2e^{2x}}{1+e^{4x}}$.

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<u>Example</u> $f(x) = (2x^2+8)^2$.

$= 4x^4 + 32x^2 + 64$.

$a(x) = x^2$.

$\cancel{b(x) = 2x^2+8}$

$\cancel{a(b(x))}$.

$b(x) = 2x+8$.

$c(x) = x^2$.

$a(b(c(x))) = (2x^2+8)^2$.

$[a(b(c(x)))]' = a'(b(c(x))) \cdot (b(c(x)))'$

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$$\Big(a(b(c(x)))\Big)' = a'\big(\,b(c(x))\,\big)\cdot\big(\,b(c(x))\,\big)' \qquad 18$$

$$= a'(b(c(x)))\; b'(c(x)) \cdot c'(x).$$

$a(x) = x^2$ $\qquad$ $a'(x) = 2x$

$b(x) = 2x+8$ $\qquad$ $b'(x) = 2$

$c(x) = x^2$ $\qquad$ $c'(x) = 2x$.

$$2(\; b(c(x)) \;)\cdot 2 \cdot 2x.$$

x^2

$2x^2+8$

$$2(2x^2+8)\cdot 4x = 8x(2x^2+8)$$

$$16x^3+64.$$

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Example $f(x) = (2x^2+8)^2$.

$= \underbrace{4x^4+32x^2+64}$.

$a(x) = x^2$.

$\cancel{b(x) = 2x^2+8}$

$b(x) = 2x+8$.

$c(x) = x^2$.

$\cancel{a(b(x))}$. diff wrt x

$16x^3+64x$.

$a(b(c(x))) = (2x^2+8)^2$.

$[a(b(c(x)))]' = a'(b(c(x))) \cdot (b(c(x)))'$

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$$(a(b(c(x))))' = a'(b(c(x))) \cdot (b(c(x)))'$$ 18

$$= a'(b(c(x))) \, b'(c(x)) \cdot c'(x).$$

$a(x) = x^2$ $\quad$ $a'(x) = 2x$

$b(x) = 2x + 8$ $\quad$ $b'(x) = 2$

$c(x) = x^2$ $\quad$ $c'(x) = 2x.$

$$2(\underbrace{b(\underbrace{c(x)}_{x^2})}_{2x^2+8}) \cdot 2 \cdot 2x.$$

$$2(2x^2+8) \cdot 4x = 8x(2x^2+8)$$

$$16x^3 + 64.$$

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<u>Example</u>

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$$f(x) = (x+1)^{1000}$$

$$a(x) = x^{1000} \qquad a'(x) = 1000\,x^{999}$$

$$b(x) = x+1 \qquad b'(x) = 1.$$

$$\left(a(b(x))\right)' = a'(b(x)) \cdot b'(x).$$

$$= 1000\,(b(x))^{999} \cdot 1.$$

$$= 1000\,(x+1)^{999}.$$

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§3.9 Derivatives of exponentials and logs

recall $f(x) = b^x$ then $f'(x) = m_b b^x$

$f(x) = e^x$ then $f'(x) = e^x$.

Thm $f(x) = b^x$ then $f'(x) = \ln(b)\, b^x$.

Proof $f(x) = b^x$ $\qquad b = e^{\ln(b)}$

$f(x) = (e^{\ln(b)})^x = e^{x\ln(b)}$

chain rule $\quad f'(x) = e^{x\ln(b)} \cdot (x\ln(b))'$

$= \ln(b) \cdot e^{x\ln(b)} = \ln(b)\, b^x \quad \square$

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Example $f(x) = 3^{4x} = (3^4)^x$.

$b^x \quad b = 3^4$.

$$f'(x) = \ln(b)\, b^x$$
$$= \ln(3^4)(3^4)^x$$
$$= 4\ln(3)\, 3^{4x}.$$

Thm $f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$.

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ex ①

$$f(x) = \log_b(x) = \frac{\ln(x)}{\ln(b)} \leftarrow \text{log conversion rule.}$$

$$f'(x) = \frac{1/x}{\ln(b)} = \frac{1}{\ln(b)\,x}.$$

② $f(x) = x\ln(x)$

$$f'(x) = (x)'\ln(x) + x(\ln(x))'$$

$$= 1\cdot\ln(x) + x\cdot\frac{1}{x}$$

$$= \ln(x) + 1.$$