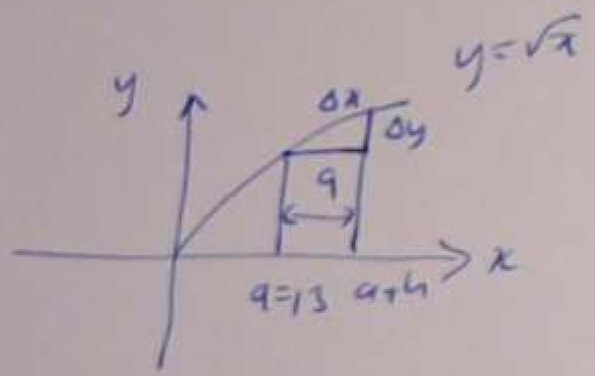
1

Q6

$f(x) = \sqrt{x}$ $\qquad a = 13 \quad h = 9$



$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

$$= \frac{f(22) - f(9)}{9} = \frac{\sqrt{22} - 3}{9}$$

Q1

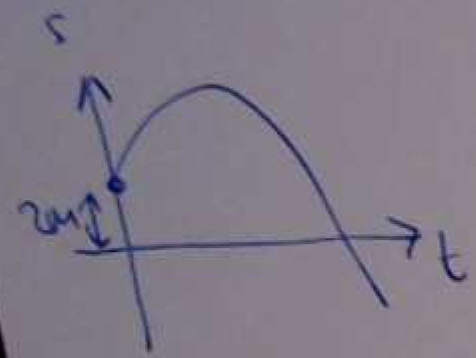
$(1, -2)$

$(3, 0)$

$$\frac{\Delta y}{\Delta x} = \frac{2}{2} = 1$$

Q2

recall motion under gravity



$$s(t) = -\tfrac{1}{2}gt^2 + v_0 t + s_0 \qquad s_0 = s(0)$$

$$s'(t) = v(t) = -gt + v_0 \qquad v_0 = v(0) = s'(0)$$

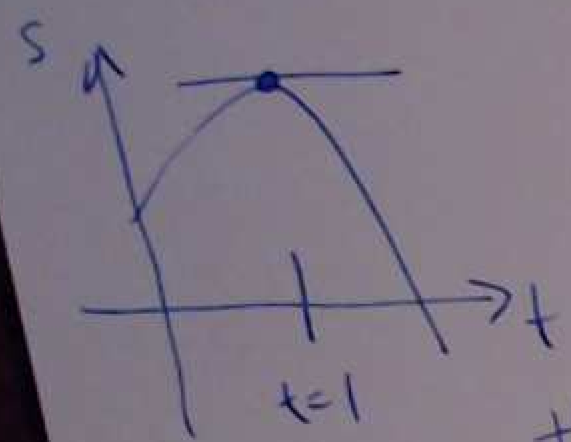
$$s''(t) = v'(t) = a(t) = -g \simeq 10\ m/s^2$$

Example throw a stone upwards at 10 m/s from a height of 2m? What is the max height?

$$s_0 = s(0) = 2$$

$$v_0 = s'(0) = 10$$

$$s(t) = -\tfrac{1}{2}10t^2 + 10t + 2$$



$$s(t) = -5t^2 + 10t + 2$$

$$s'(t) = -10t + 10$$

the max height is when

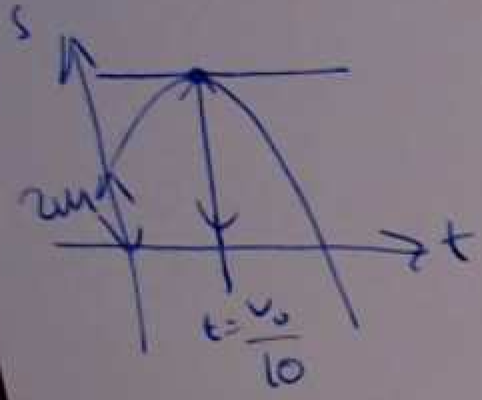
$$s'(t) = 0 = -10t + 10$$

$$10t = 10 \qquad t = 1$$

$$s(1) = -5 + 10 + 2 = 7\,m.$$

④

Q: if I can throw the stone 10m high how fast can I throw it?



$$s(t) = -\frac{1}{2}\underbrace{g}_{\approx 10}t^2 + \underbrace{v_0}_{\text{don't know}}t + \underbrace{s_0}_{2}$$

$$s(t) = -5t^2 + v_0 t + 2$$

$$s'(t) = -10t + v_0 = 0$$

$$10t = v_0 \qquad t = \frac{v_0}{10}$$

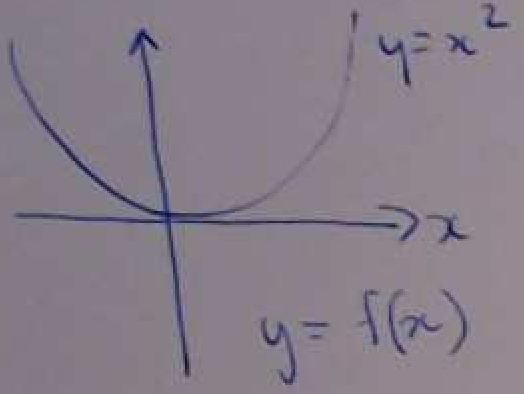
$$s\left(\frac{v_0}{10}\right) = -5\frac{v_0^2}{100} + v_0\frac{v_0}{10} + 2 = 10$$

$$v_0^2\underbrace{\left(\frac{1}{10} - \frac{1}{20}\right)}_{\frac{1}{20}} = 8$$

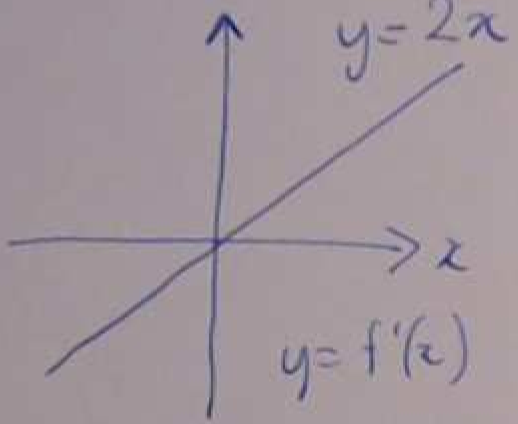
$$v_0^2 = 8 \times 20 = 160$$

$$\boxed{v_0 = \sqrt{160} = 4\sqrt{10}}$$

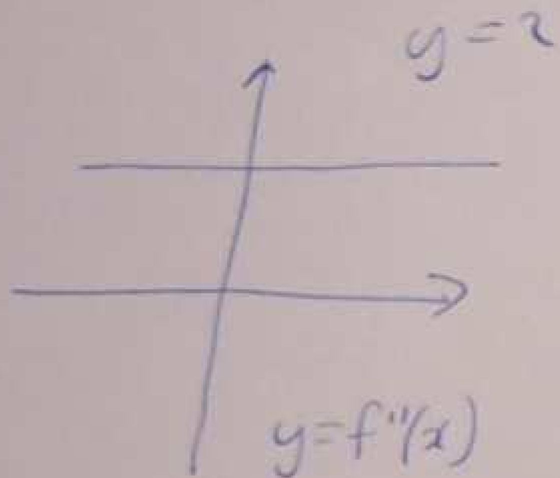
§ 3.5 Higher derivatives



"distance"



"velocity"



"acceleration"

Examples $f(x) = xe^x$

$f'(x) =$ $\qquad xe^x$

product rule $(fg)' = f'g + fg'$

$$f(x) = \underbrace{x}_{f}\,\underbrace{e^{x}}_{g}$$

$$(fg)' = f'g + fg'$$

product rule

power of x

$$(x^n)' = nx^{n-1}$$

exp.

$$(e^x)' = e^x$$

$$f'(x) = f'g + fg'$$

$$(x)'e^x + x(e^x)'$$

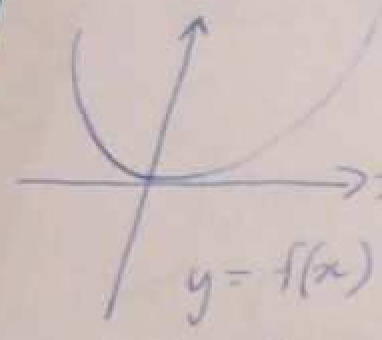
$$1 \cdot e^x + x \cdot e^x$$

$$= e^x + x \cdot e^x$$

$$= e^x(1+x)$$

recal

§3.5



"distance"

Examples $f(x)$

$f'(x) =$

product rul

$$f(x) = xe^{x}$$

$$f'(x) = e^{x} + xe^{x}.$$

$$f''(x) = (e^{x})' + (xe^{x})'.$$

$$= e^{x} + (x)'e^{x} + x(e^{x})'.$$

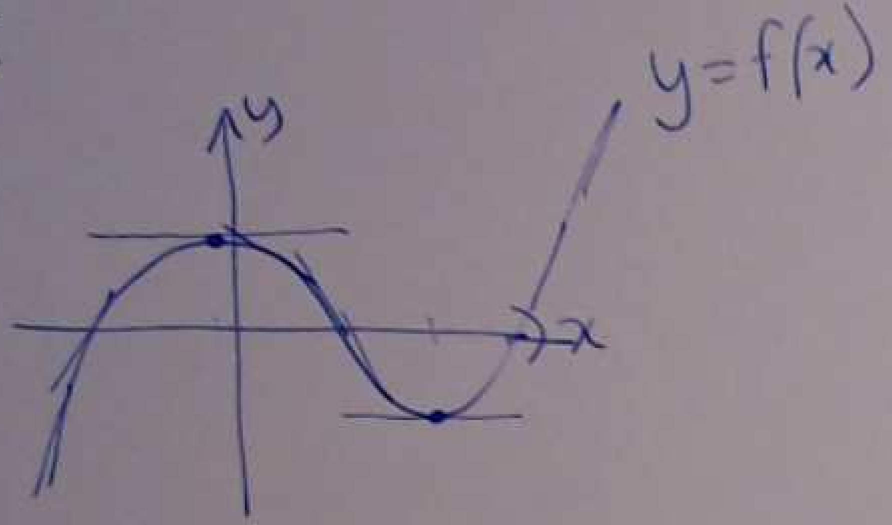
$$= e^{x} + e^{x} + xe^{x}$$

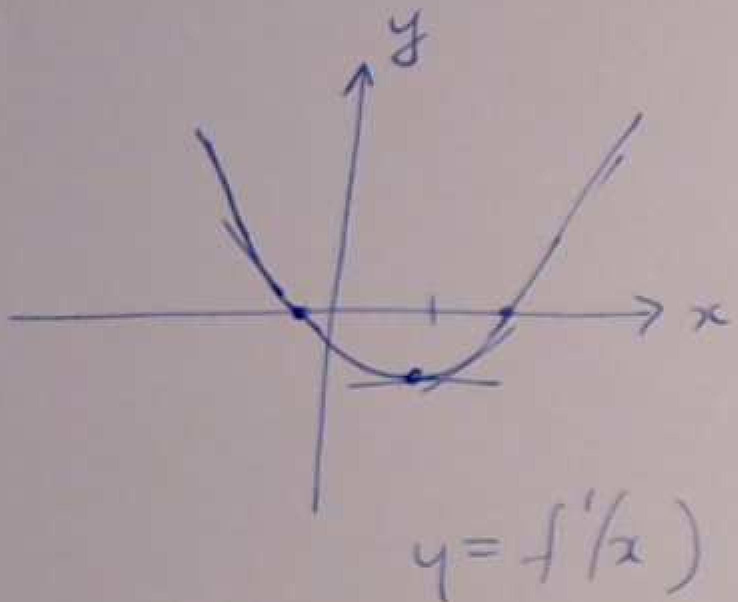
$$= 2e^{x} + xe^{x}.$$

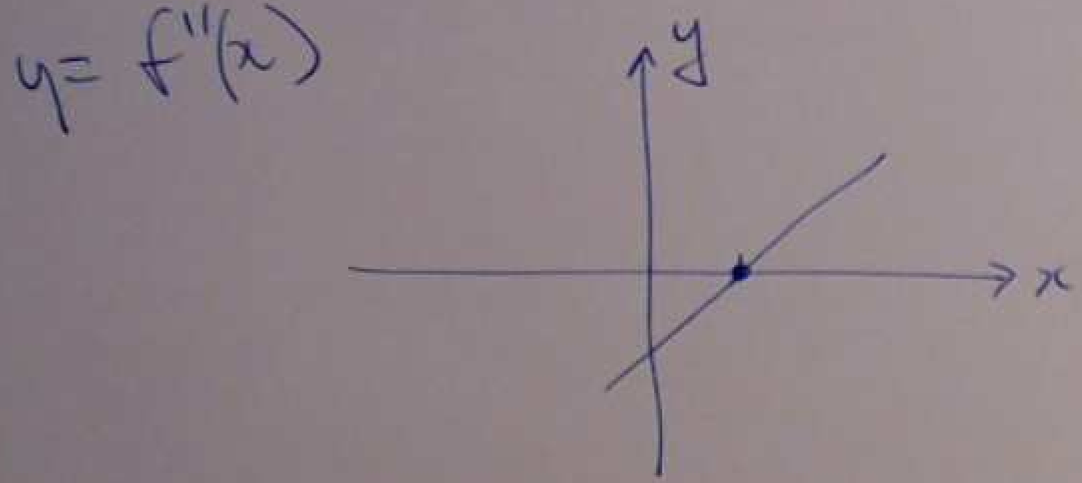
$$f'''(x) = 2e^{x} + (x)'e^{x} + x(e^{x})'$$

$$f^{(3)}(x) = 3e^{x} + xe^{x}$$

Example





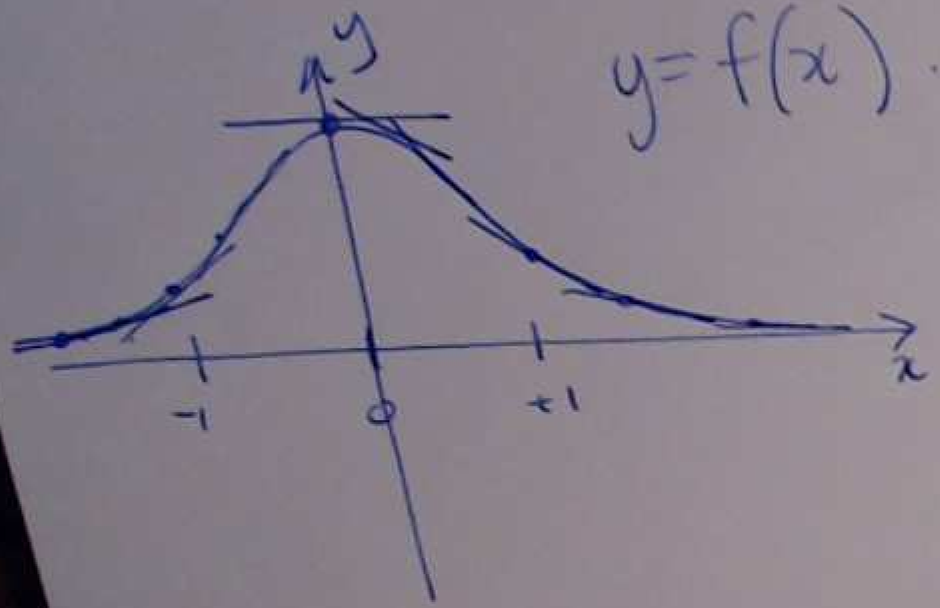


polynomial: $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$.

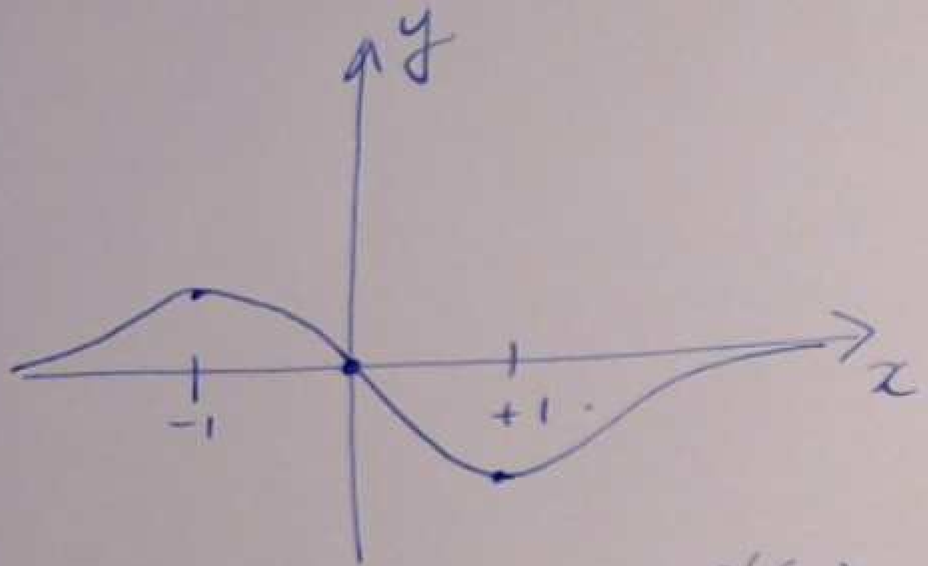
exponential function: e^x.

$$f(x) = e^x$$
$$f'(x) = e^x$$
$$f''(x) = e^x$$
$$f^{(3)}(x) = e^x$$
$$\vdots$$
$$f^{(n)}(x) = e^x$$
$$\vdots$$

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
$$p'(x) = a_1 + 2a_2 x + \cdots + a_n n x^{n-1}$$
$$\vdots$$
$$p^{(n)}(x) = a_n n(n-1) \cdots 2 \cdot 1$$
$$p^{(n+1)}(x) = 0$$

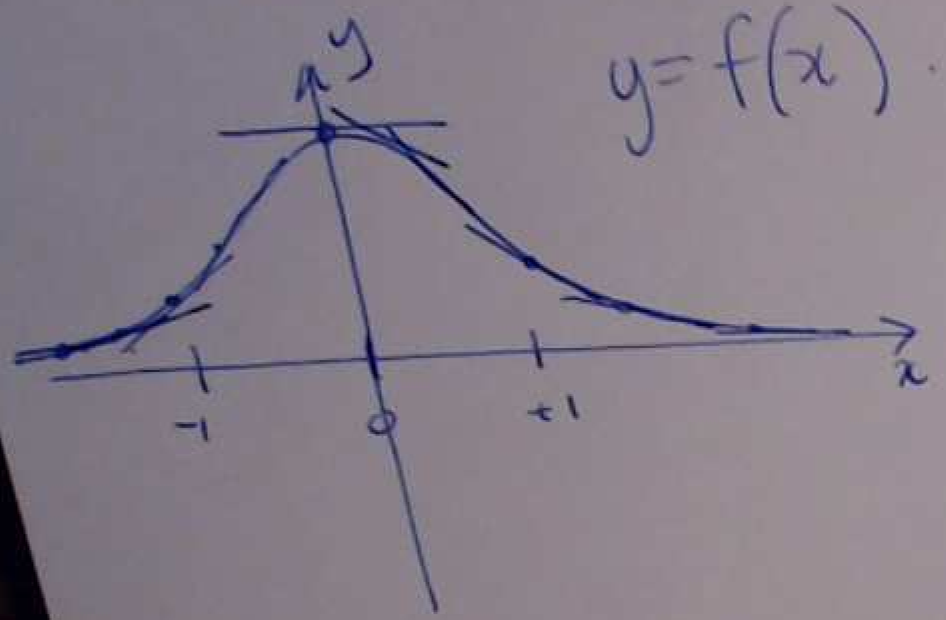


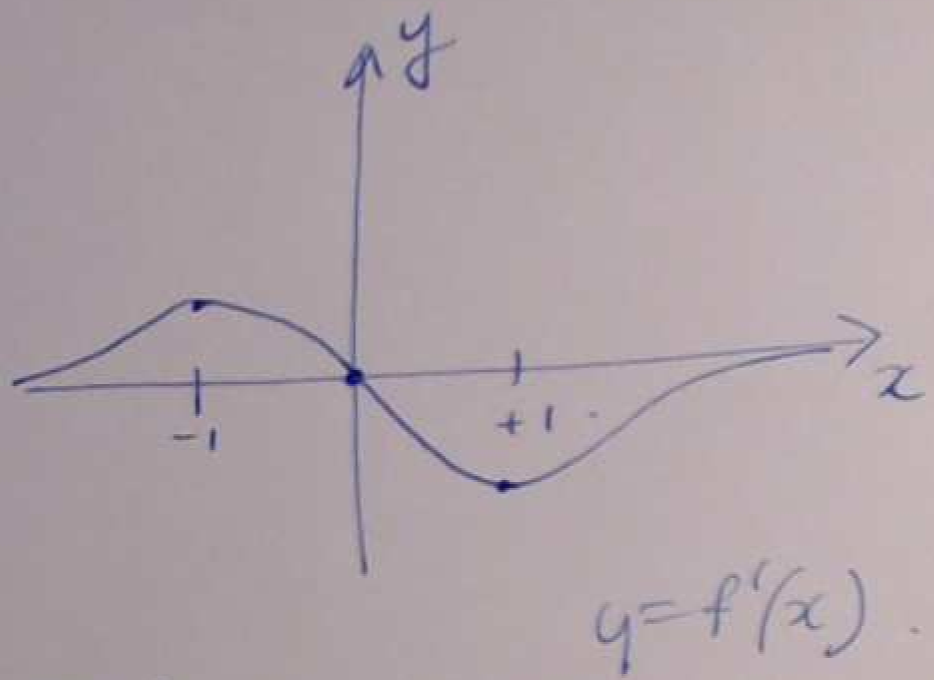
$y = f(x)$.



$y = f'(x)$.

$f(x)$		$f'(x)$
increasing	$\Leftrightarrow$	positive.
decreasing	$\Leftrightarrow$	negative.





$f(x)$ $\qquad$ $f'(x)$

increasing $\iff$ positive.

decreasing $\iff$ negative.

§3.6 Trig functions

Thm
$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x)$$

$$\frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

recall $\sin(A+B) = \sin A \cos B + \cancel{\sin}\cos A \sin B$

Proof $(\sin x)$

$$\frac{d}{dx}\left(\sin(x)\right) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

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$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad f(x) = \sin(x)$$

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$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

$$= \lim_{h\to 0} \frac{\sin x \cosh + \cos x \sin h - \sin x}{h}.$$

$$= \lim_{h\to 0} \sin x \frac{(\cos h - 1)}{h} + \cos x \cdot \frac{\sin h}{h}.$$

$$= \sin x \underbrace{\boxed{\lim_{h\to 0} \frac{\cos h - 1}{h}}}_{=0} + \cos x \underbrace{\boxed{\lim_{h\to 0} \frac{\sin h}{h}}}_{=1}$$

$$= \cos x \cdot \square$$

recall

Proof (sin

$\frac{d}{dx}$

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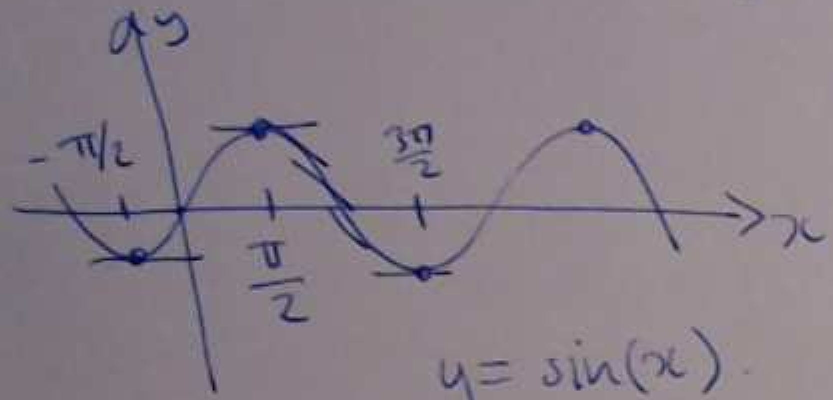
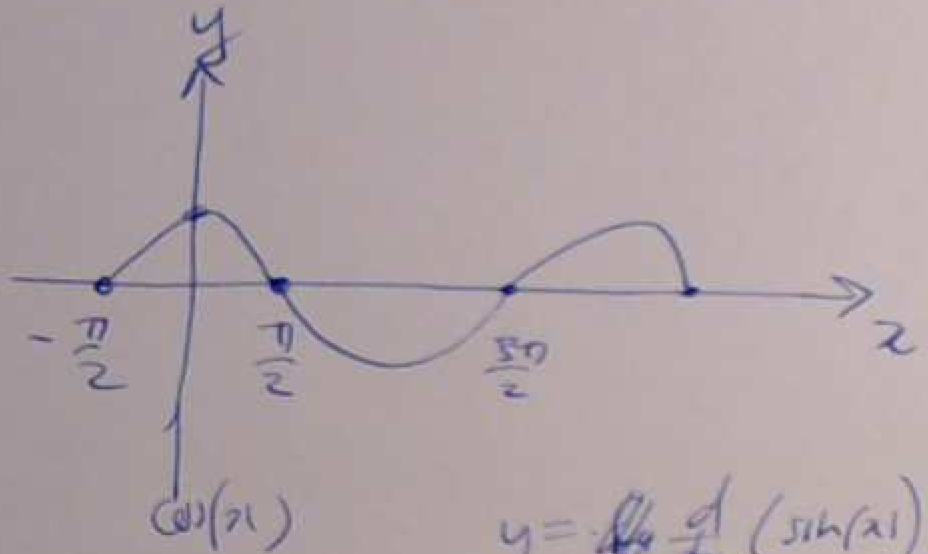
$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

Q: can this be right?





Example $f(x) = x\sin(x)$ product rule

$$f'(x) = (x)'\sin(x) + x\cdot(\sin(x))' \qquad (fg)' = f'g + fg'$$

$$= \sin(x) + x\cos(x)$$

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Thm $\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad \frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ $\qquad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$

Proof ($\tan x$).

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \leftarrow \text{quotient rule.}$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

$$= \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x}$$

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$$= \frac{\cos(x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x}.$$

$$= \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x}.$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. \quad \square.$$

Summary $\frac{d}{dx}(\tan x) = \sec^2 x.$

Thm

Proof (tan

$$\frac{d}{dx}(\tan x)$$

$$= \frac{(\cos x)(\sin x)}{\cos^2 x}$$

Example $\dfrac{d}{dx}\left(\dfrac{e^x}{\tan x}\right) \leftarrow$ quotient rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$= \frac{\tan x\,(e^x)' - e^x(\tan x)'}{(\tan x)^2}$$

$$= \frac{\tan x \cdot e^x - e^x \cdot \sec^2 x}{\tan^2 x}$$

summary

observation.

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f^{(3)}(x) = e^x$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = \sin(x)$

$f^{(5)}(x) = \cos(x)$

$\vdots$

Q: can you find a function s.t. $f^{(3)}(x) = f(x)$

$(\neq f'(x), f''(x))$

$f^{(2)}(x) = -f(x)$?

Example

$= \tan x$

$= \tan x$

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§ 3.7 Chain rule

notation

composition of functions $f(g(x)) = (f \circ g)(x)$

Examples 1 e^{4x} $4x = g(x)$

$e^x = f(x)$

$f(g(x))$ $e^{(4x)} = e^{4x}$

2 $\sin^2 x$ $f(g(x))$ $g(x) = \sin(x)$

$\quad = (\sin x)^2$ $f(x) = x^2$

$f(g(x)) = (\sin x)^2 = \sin^2 x$

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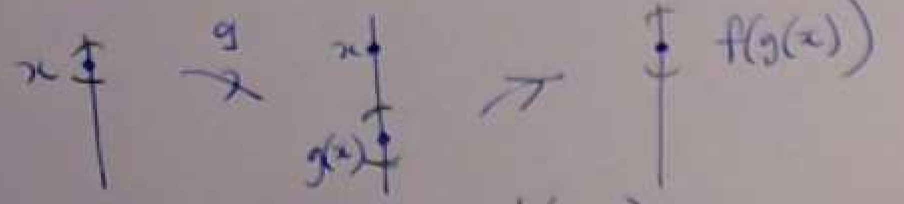
Thm Chain rule If f and g are differentiable then $f \circ g$ is differentiable,

$$\left(f(g(x)) \right)' = f'(g(x)) \cdot g'(x)$$

WARNING
$\neq f'(x) g'(x)$ ✗

mnemonic: $[f(g(x))]' = \text{outside}'(\text{inside}) \cdot \text{inside}'$

note $f(g(x))$ $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$



$g'(x)$ stretch factor

$f'(g(x))$ stretch factor

overall stretch $= g'(x) \times f'(g(x))$

Examples ① $h(x) = e^{4x} = f(g(x))$

$g(x) = 4x$

$f(x) = e^{x}$

$h'(x) = \left(f(g(x))\right)'$

$= f'(g(x)) \cdot g'(x)$

$g'(x) = 4$

$f'(x) = e^{x}$

$= e^{g(x)} \cdot 4$

$= e^{4x} \cdot 4 = 4e^{4x}$

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(21)

Exercise $e^{4x} = (e^x)^4$ $f(g(x))$

$g(x) = e^x$

$f(x) = x^4$

check this gives the same answer.

Example (2) $h(x) = \sin^2(x) = (\sin(x))^2 = f(g(x))$.

$f(x) = x^2$

$g(x) = \sin(x)$.

$h'(x) = f'(g(x)) \cdot g'(x)$

$= 2g(x) \cdot \cos(x)$

$= 2\sin(x)\cos(x)$.

$f'(x) = 2x$

$g'(x) = \cos(x)$

Examples

$h'(x) =$

$= f'(g($

$= e^{g(x)}$

$= e^{4x}$

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③

$$f(x) = \sqrt{1+\sqrt{1+x}}$$

$$= \sqrt{\left(1+\sqrt{1+x}\right)}.$$

$$f'(x) = a'\left(b(x)\right).b'(x)$$

$$= \frac{1}{2\sqrt{b(x)}}.b'(x).$$

$$= \frac{1}{2\sqrt{1+\sqrt{1+x}}}.b'(x)$$

$$\left(a(b(x))\right)' = a'(b(x)).b'(x)$$

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$$a(x) = \sqrt{x} = x^{1/2}.$$

$$b(x) = 1+\sqrt{1+x}.$$

$$a'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$$

$$b'(x) = ?$$

$$= \left(\sqrt{1+x}\right)'.$$

$$= c(d(x))$$

$c(x) = \sqrt{x}$

$d(x) = 1+x.$

$$b'(x) = c'(d(x)).d'(x)$$

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$b(x) = 1 + \sqrt{1+x}$

$b'(x) = \left(\sqrt{1+x}\right)'$

$c(x) = \sqrt{x} = x^{1/2}$ $\quad c'(x) = \frac{1}{2}x^{-1/2}$

$d(x) = 1+x$ $\quad d'(x) = 1$

$b'(x) = c'(d(x)) \cdot d'(x) = \frac{1}{2}(d(x))^{-1/2} \cdot 1 = \frac{1/2}{\sqrt{1+x}}$

$f'(x) = \frac{1}{2\sqrt{1+\sqrt{1+x}}} \cdot \frac{1}{2\sqrt{1+x}}$

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WW 3.3 Q9

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$$f(x) = \underbrace{7x R(x)} + \underbrace{7Q(x) R(x)} + Q(x)$$

$Q'(x) = -R(x)$

$R'(x) = P(x)$

product rule.

$(ab)' = a'b + ab'$.

$$f'(x) = (7x)' R(x) + 7x (R(x))' + 7(Q(x))' R(x) + 7Q(x)(R(x))' + Q'(x)$$

$$= 7R(x) + 7x P(x) + 7(-R(x))R(x) + 7Q(x) P(x) + -R(x)$$

$$= 7R(x) + 7x P(x) - 7(R(x))^2 + 7Q(x) P(x) - R(x).$$