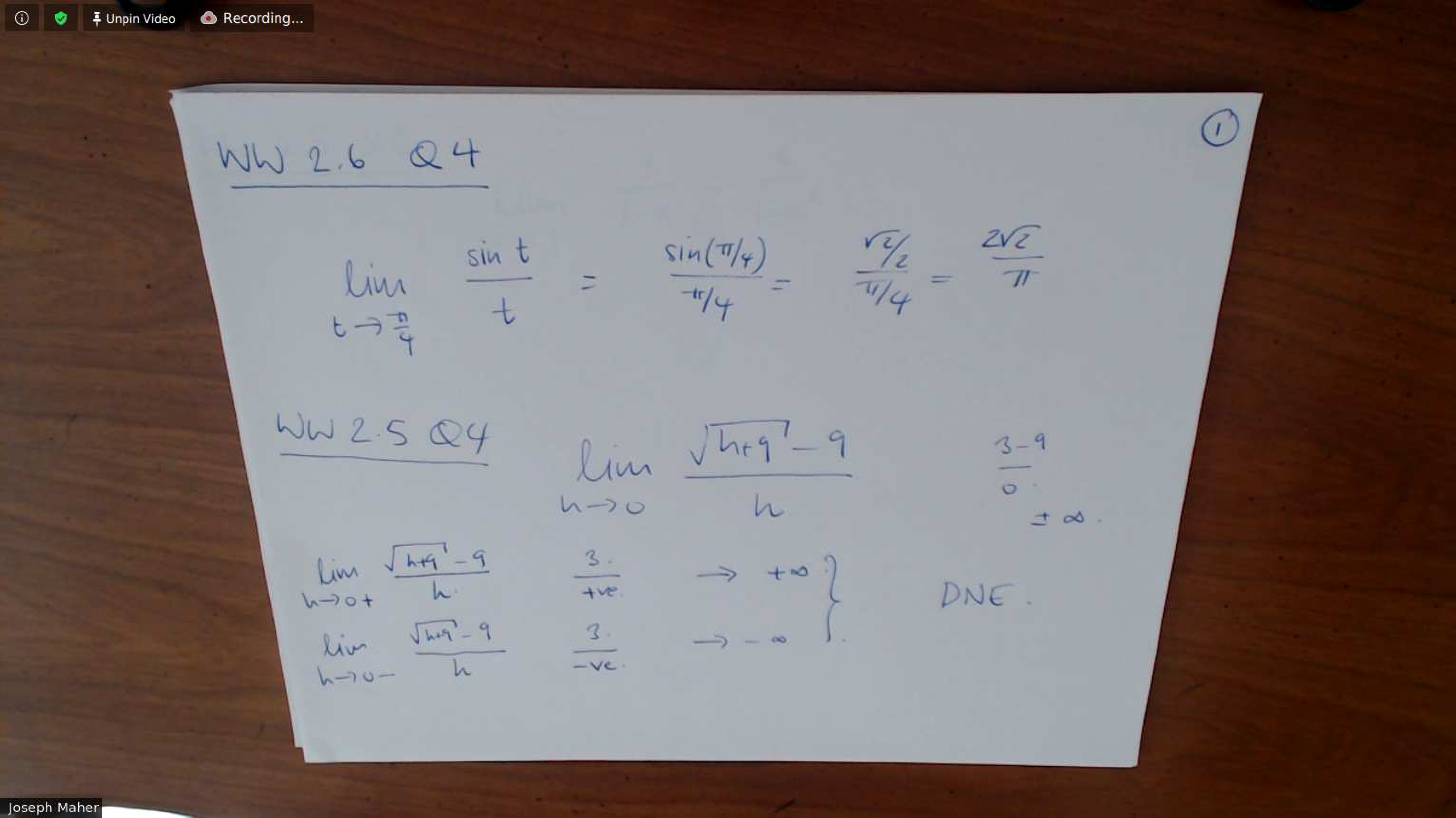


①

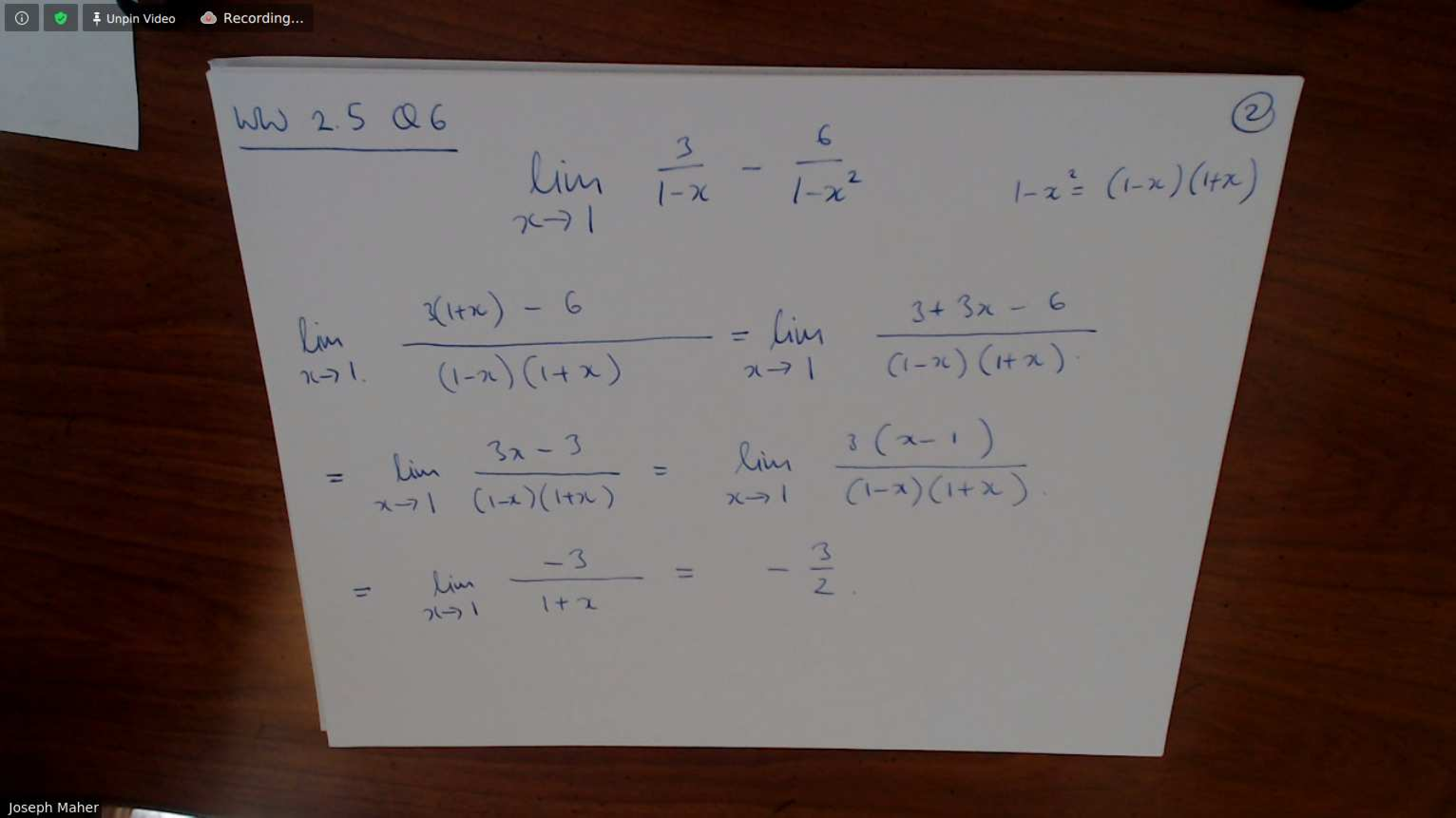
WW 2.6 Q4

$$\lim_{t\to\frac{\pi}{4}} \frac{\sin t}{t} = \frac{\sin(\pi/4)}{\pi/4} = \frac{\sqrt{2}/2}{\pi/4} = \frac{2\sqrt{2}}{\pi}$$

WW 2.5 Q4

$$\lim_{h\to 0} \frac{\sqrt{h+9}-9}{h} \qquad \frac{3-9}{0} \qquad \pm\infty.$$

$$\left.\begin{array}{lll} \lim\limits_{h\to 0+} \frac{\sqrt{h+9}-9}{h} & \frac{3}{+ve} & \to +\infty \\ \lim\limits_{h\to 0-} \frac{\sqrt{h+9}-9}{h} & \frac{3}{-ve} & \to -\infty \end{array}\right\} \quad \text{DNE}.$$



WW 2.5 Q6

$$\lim_{x\to 1} \frac{3}{1-x} - \frac{6}{1-x^2}$$

$1-x^2 = (1-x)(1+x)$

$$\lim_{x\to 1.} \frac{3(1+x) - 6}{(1-x)(1+x)} = \lim_{x\to 1} \frac{3+3x-6}{(1-x)(1+x)}.$$

$$= \lim_{x\to 1} \frac{3x-3}{(1-x)(1+x)} = \lim_{x\to 1} \frac{3(x-1)}{(1-x)(1+x)}.$$

$$= \lim_{x\to 1} \frac{-3}{1+x} = -\frac{3}{2}.$$

WW 2.5 Q7

③

$$\lim_{x\to 1}\frac{1-x^{-1}}{x-2+x^{-1}}\begin{matrix}\times x\\ \times x\end{matrix}$$

plug in 1 $\frac{0}{0}$.

indeterminate form.

$$\lim_{x\to 1}\frac{x-1}{x^2-2x+1}=\lim_{x\to 1}\frac{x-1}{(x-1)(x-1)}=\lim_{x\to 1}\frac{1}{x-1}\quad "\frac{1}{0}"$$

not indeterminate.

means → vertical asymptote $\pm\infty$ DNE.

$$\left.\begin{aligned}\lim_{x\to 1^+}\frac{1}{x-1}&=+\infty\\ \lim_{x\to 1^-}\frac{1}{x-1}&=-\infty.\end{aligned}\right\}\quad \lim_{x\to 1}\frac{1}{x-1}\ \text{DNE.}$$

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WW 2.5 Q10

$$\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{4(x-a)}$$

$\frac{0}{0}$ indeterminate form.

$$\lim_{x\to 0} \frac{\cancel{(\sqrt{x}-\sqrt{a})}}{4(\sqrt{x}+\sqrt{a})\cancel{(\sqrt{x}-\sqrt{a})}}$$

difference of two squares

$x-a = (\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})$

$a^2-b^2 = (a-b)(a+b)$

$$\lim_{x\to 0} \frac{1}{4(\sqrt{x}+\sqrt{a})} = \frac{1}{4\sqrt{a}}.$$

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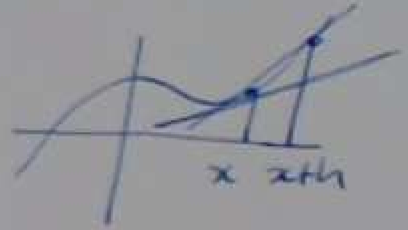
recall $f(x)$

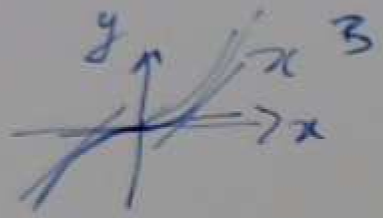
$$\boxed{\text{derivative definition} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example $f(x) = x^3$

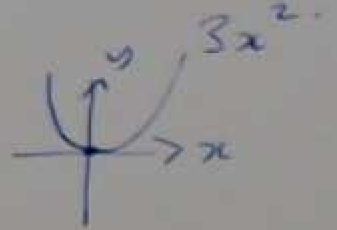


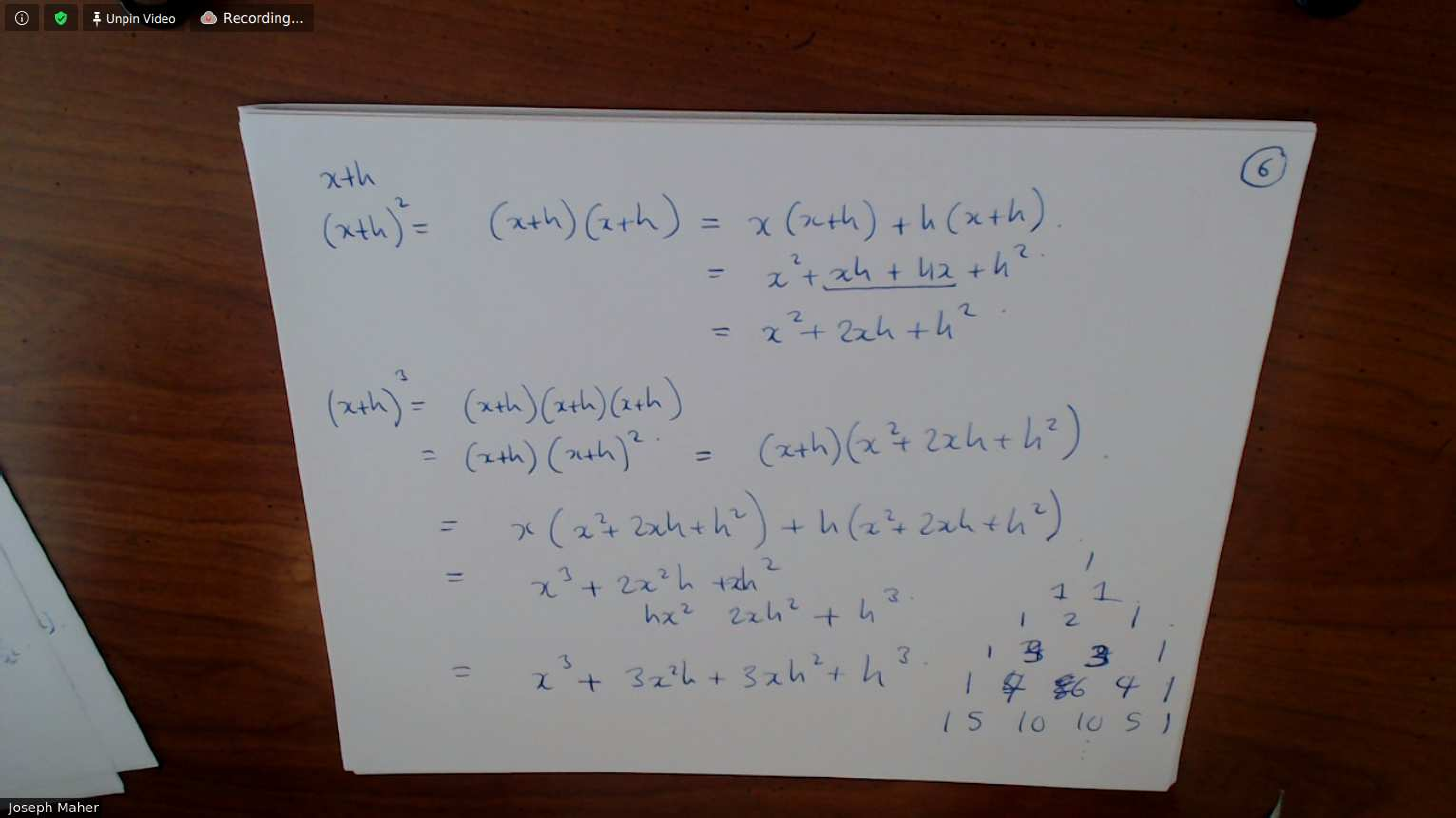
$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$



$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$



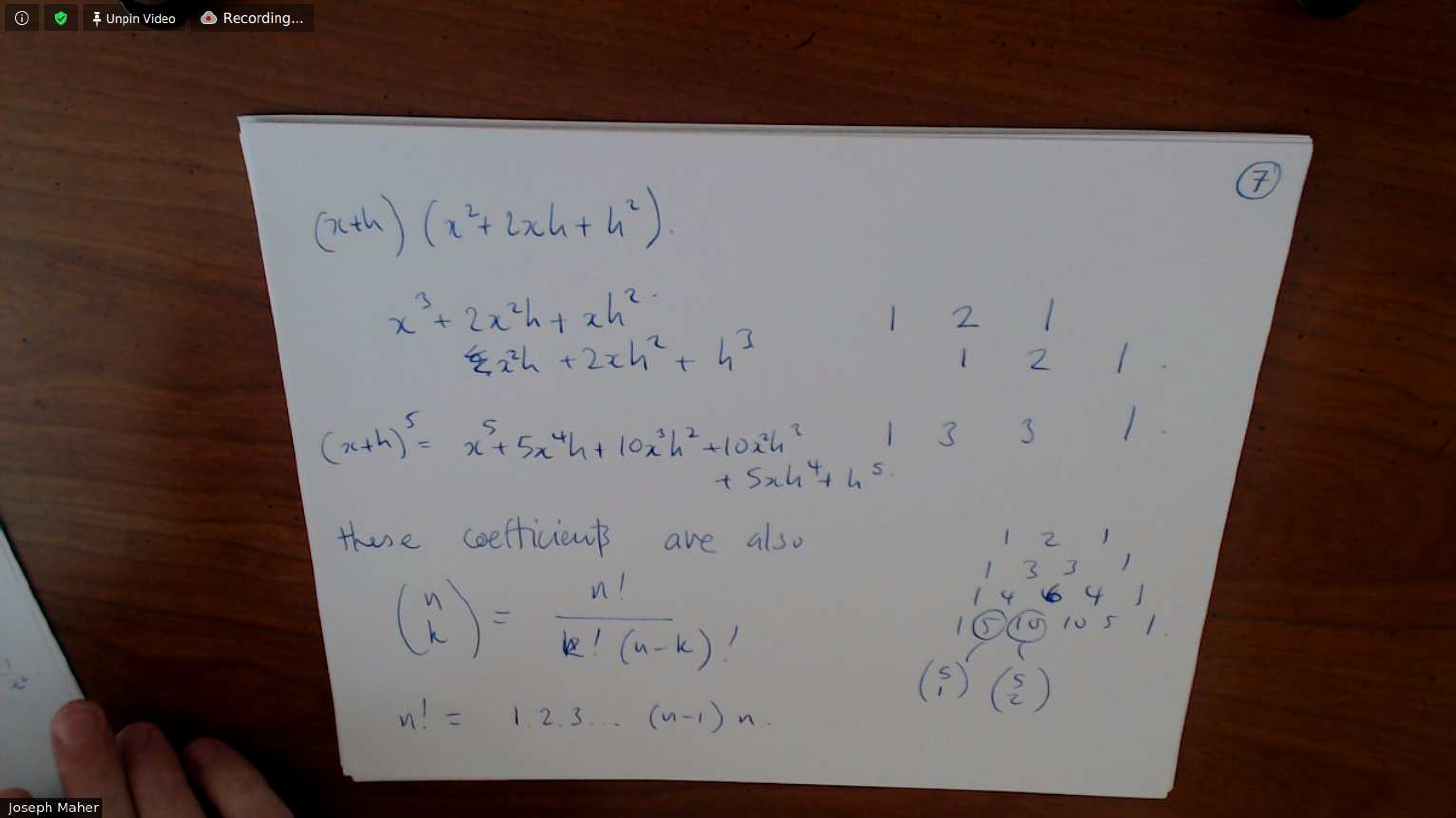


$x+h$

$$(x+h)^2 = (x+h)(x+h) = x(x+h) + h(x+h).$$
$$= x^2 + \underline{xh + hx} + h^2.$$
$$= x^2 + 2xh + h^2.$$

$$(x+h)^3 = (x+h)(x+h)(x+h)$$
$$= (x+h)(x+h)^2. = (x+h)(x^2 + 2xh + h^2).$$
$$= x(x^2 + 2xh + h^2) + h(x^2 + 2xh + h^2).$$
$$= x^3 + 2x^2h + xh^2$$
$$hx^2 \quad 2xh^2 + h^3.$$
$$= x^3 + 3x^2h + 3xh^2 + h^3.$$

1
1 1
1 2 1.
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1



$(x+h)(x^2+2xh+h^2)$.

$x^3+2x^2h+xh^2$.   1   2   1

$x^2h+2xh^2+h^3$ 1   2   1.

$(x+h)^5 = x^5+5x^4h+10x^3h^2+10x^2h^3$
$+5xh^4+h^5$.   1   3   3   1.

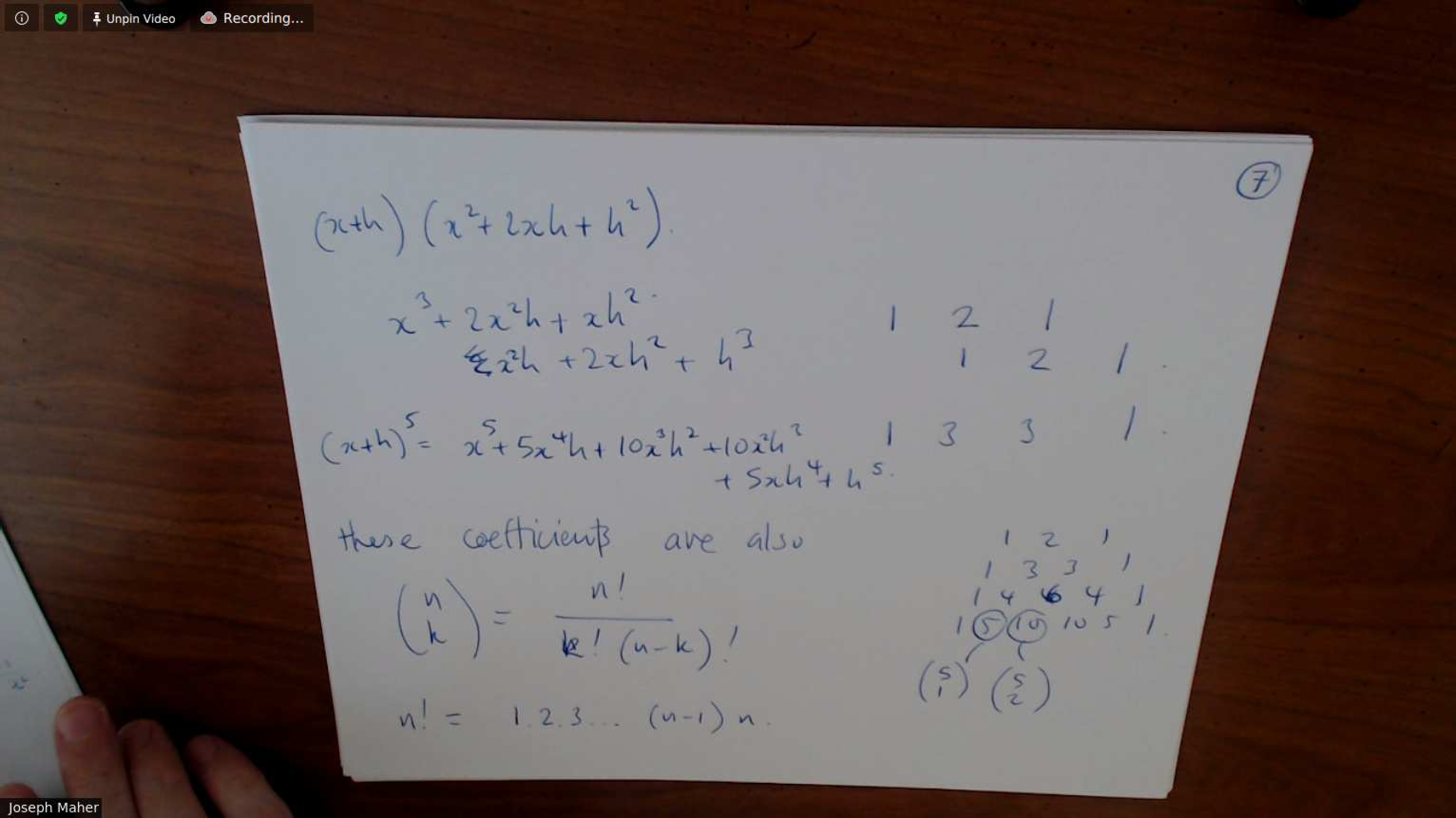
these coefficients are also

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$n! = 1.2.3\ldots(n-1)n$.

1 2 1
1 3 3 1
1 4 6 4 1
1 $\enclose{circle}{5}$ $\enclose{circle}{10}$ 10 5 1.

$\binom{5}{1}$ $\binom{5}{2}$



$(x+h)\left(x^2+2xh+h^2\right)$.

$x^3+2x^2h+xh^2$.   1 2 1

$\cancel{2}x^2h+2xh^2+h^3$ 1 2 1.

$(x+h)^5 = x^5+5x^4h+10x^3h^2+10x^2h^3$
$+5xh^4+h^5$.   1 3 3 1.

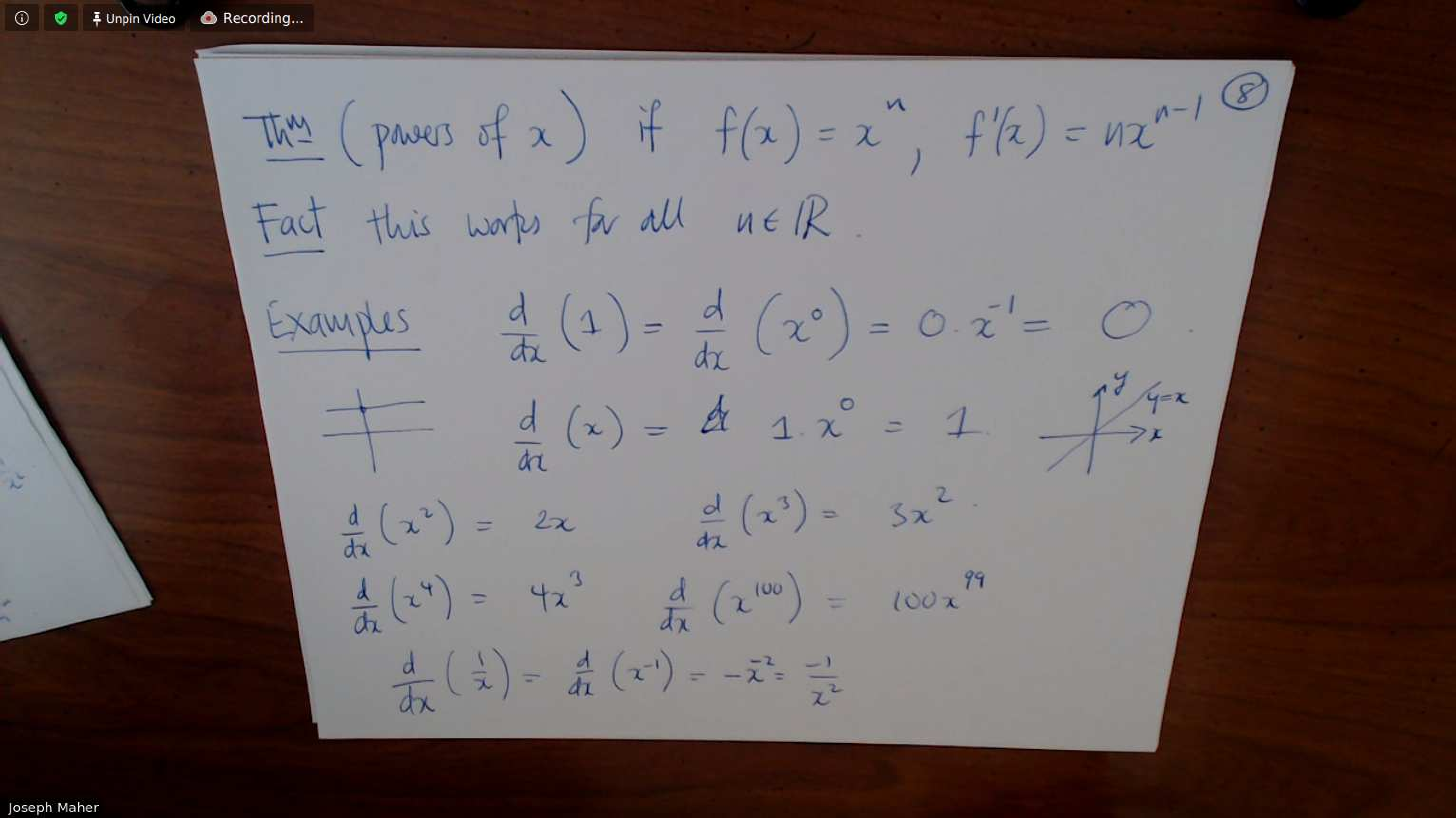
these coefficients are also

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$n! = 1.2.3\ldots(n-1)\,n$.

$$\begin{array}{c} 1\quad 2\quad 1 \\ 1\quad 3\quad 3\quad 1 \\ 1\quad 4\quad 6\quad 4\quad 1 \\ 1\quad \boxed{5}\quad \boxed{10}\quad 10\quad 5\quad 1. \end{array}$$

$\binom{5}{1}$ $\binom{5}{2}$



Thm (powers of x) if $f(x) = x^n$, $f'(x) = nx^{n-1}$

Fact this works for all $n \in \mathbb{R}$.

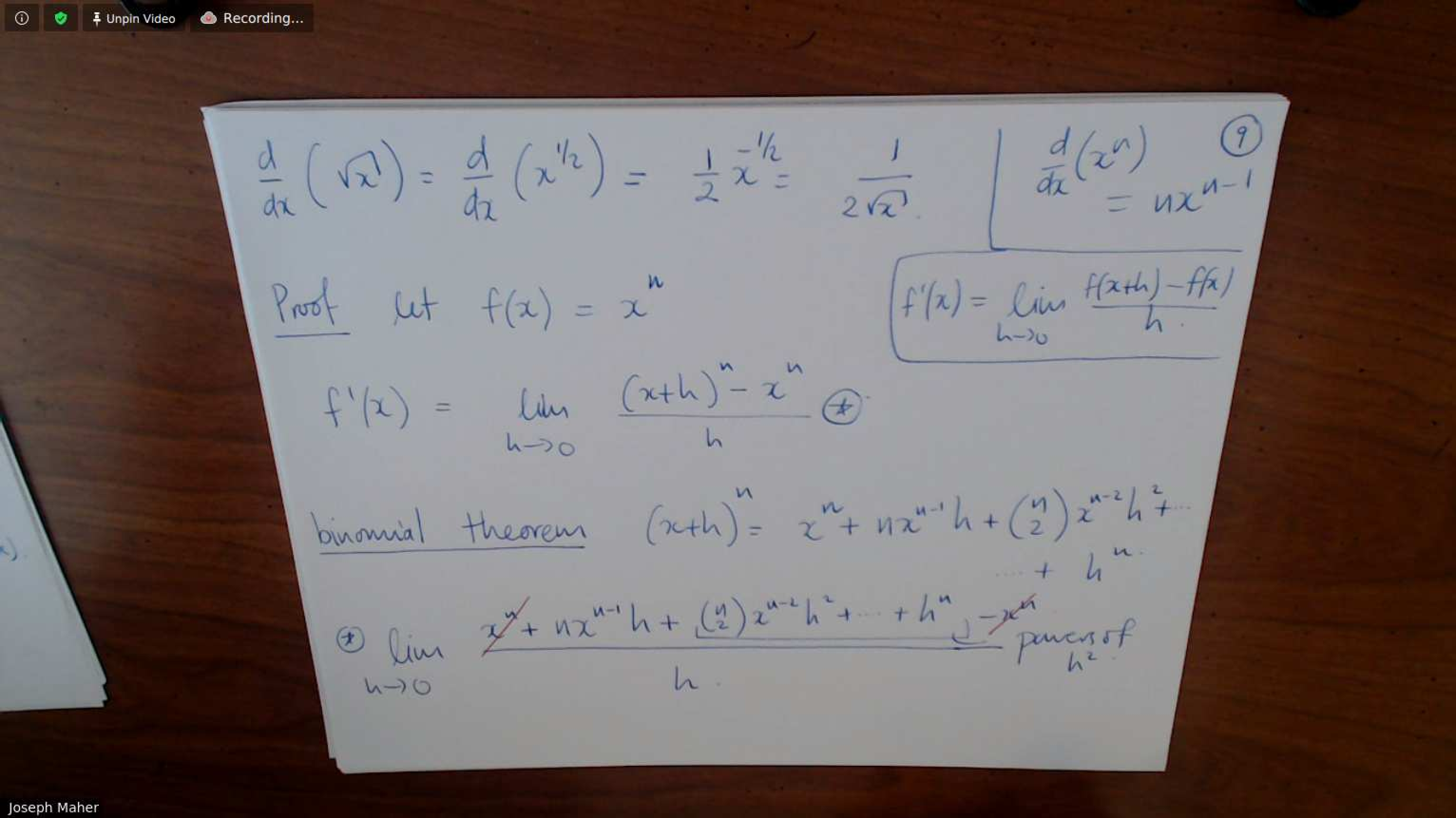
Examples $\frac{d}{dx}(1) = \frac{d}{dx}(x^0) = 0 \cdot x^{-1} = 0$.

$\frac{d}{dx}(x) = \cancel{d}\ 1 \cdot x^0 = 1$

$\frac{d}{dx}(x^2) = 2x$ $\qquad$ $\frac{d}{dx}(x^3) = 3x^2$.

$\frac{d}{dx}(x^4) = 4x^3$ $\qquad$ $\frac{d}{dx}(x^{100}) = 100x^{99}$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2}$



⑨

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.}$$

Proof let $f(x) = x^n$

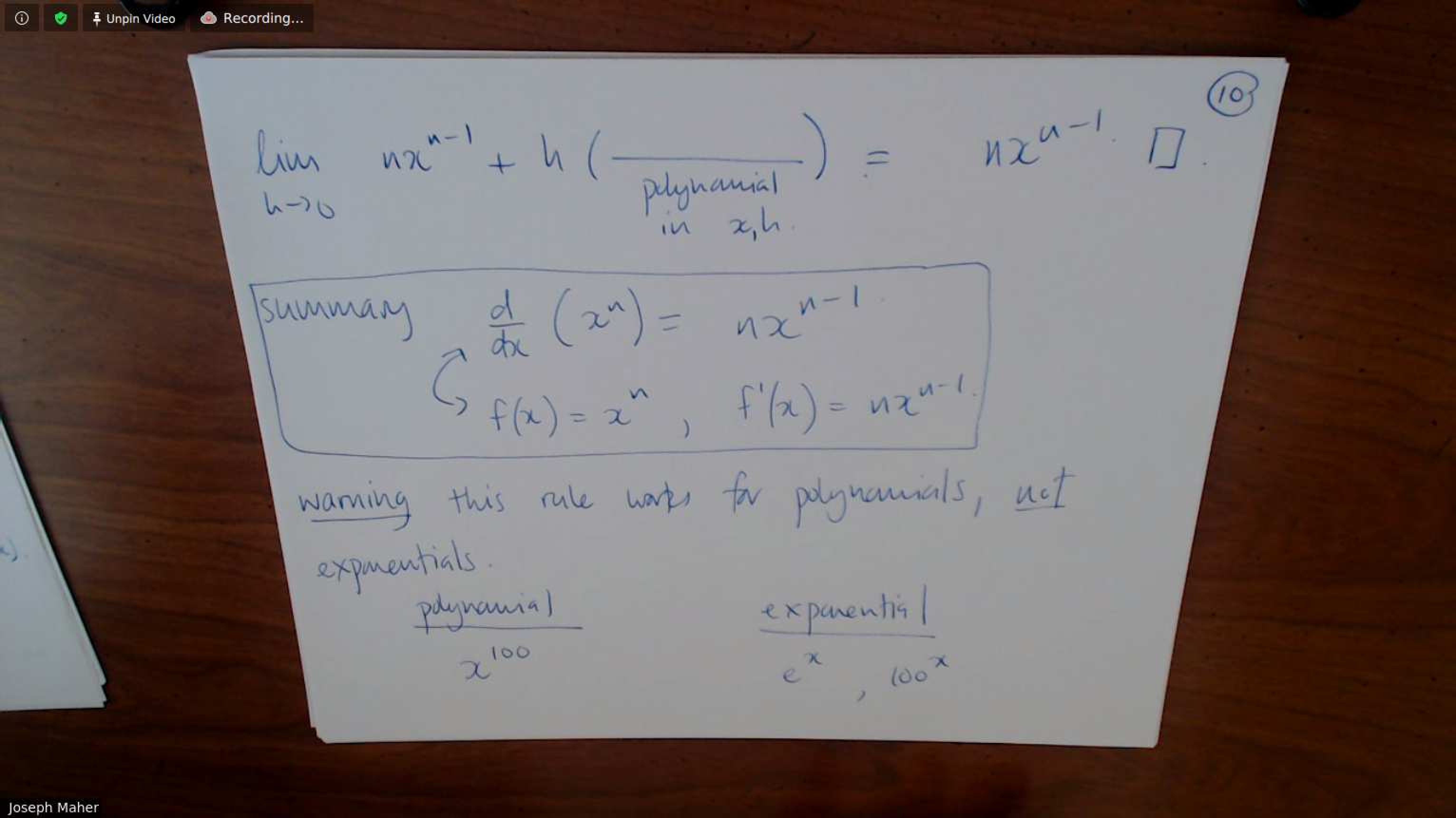
$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \circledast.$$

binomial theorem $(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots$

$\cdots + h^n.$

$$\circledast \lim_{h \to 0} \frac{\cancel{x^n} + nx^{n-1}h + \underbrace{\binom{n}{2}x^{n-2}h^2 + \cdots + h^n}_{} - \cancel{x^n}}{h}.$$

powers of h^2.



$$\lim_{h\to 0} nx^{n-1} + h\Big(\underset{\substack{\text{polynomial}\\ \text{in } x,h.}}{\underline{\qquad\qquad}}\Big) = nx^{n-1}. \quad \square.$$

summary $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

$f(x) = x^n$, $f'(x) = nx^{n-1}$.

<u>warning</u> this rule works for polynomials, <u>not</u> exponentials.

polynomial	exponential
x^{100}	e^x, 100^x

other useful rules

Thm (linearity) If f and g are differentiable functions then $f+g$ is differentiable with derivative $f'+g'$.

$(f+g)'$

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}.$$

If k is constant $(kf)' = k f'$

$$\frac{d}{dx}(kf) = k\frac{df}{dx}.$$

⑫

f

$f(x)$

<u>Proof</u> (follows from the limit laws).

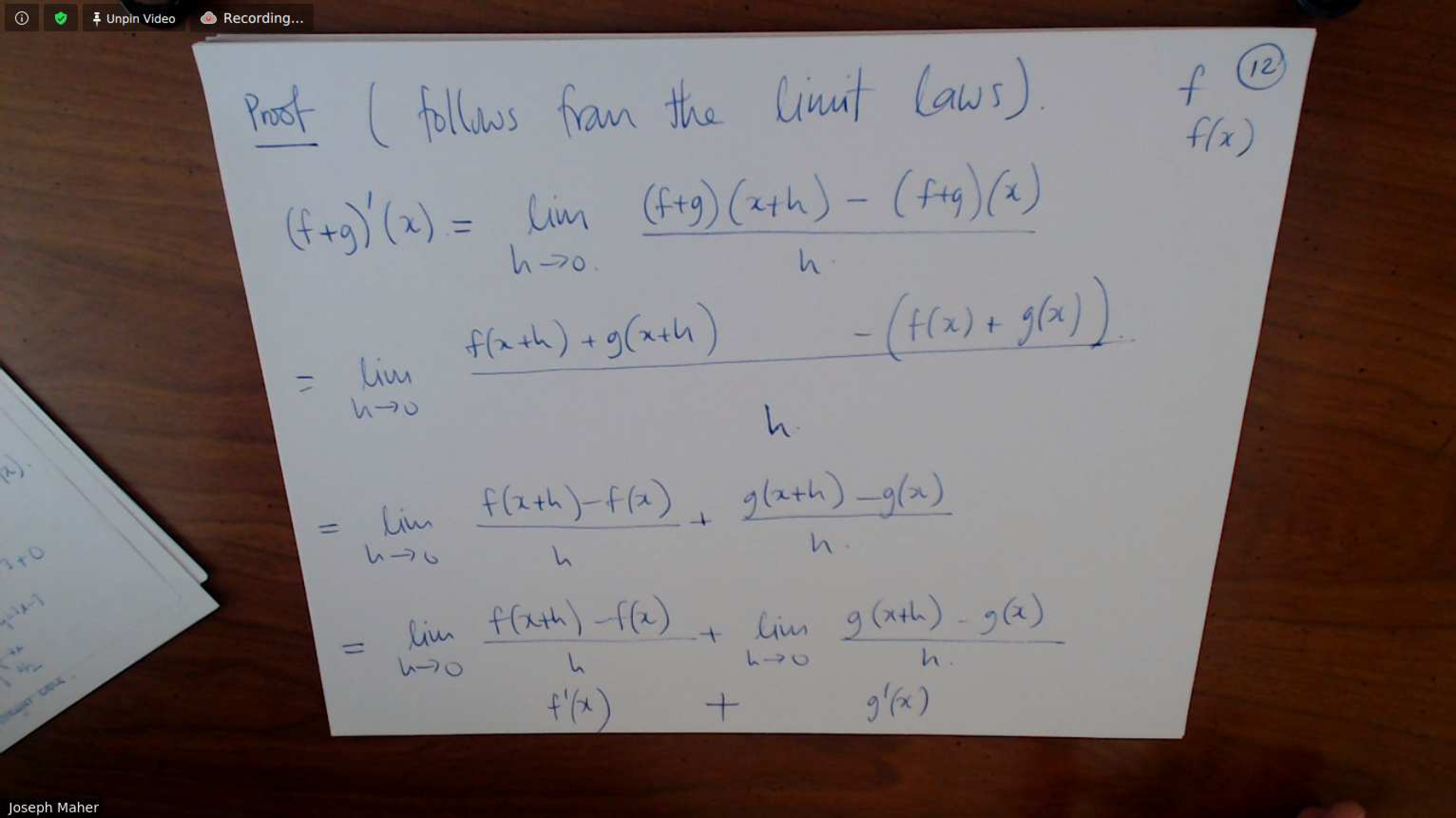
$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) \qquad - \big(f(x) + g(x)\big)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$f'(x) \qquad + \qquad g'(x)$$



⑫

f

$f(x)$

Proof (follows from the limit laws).

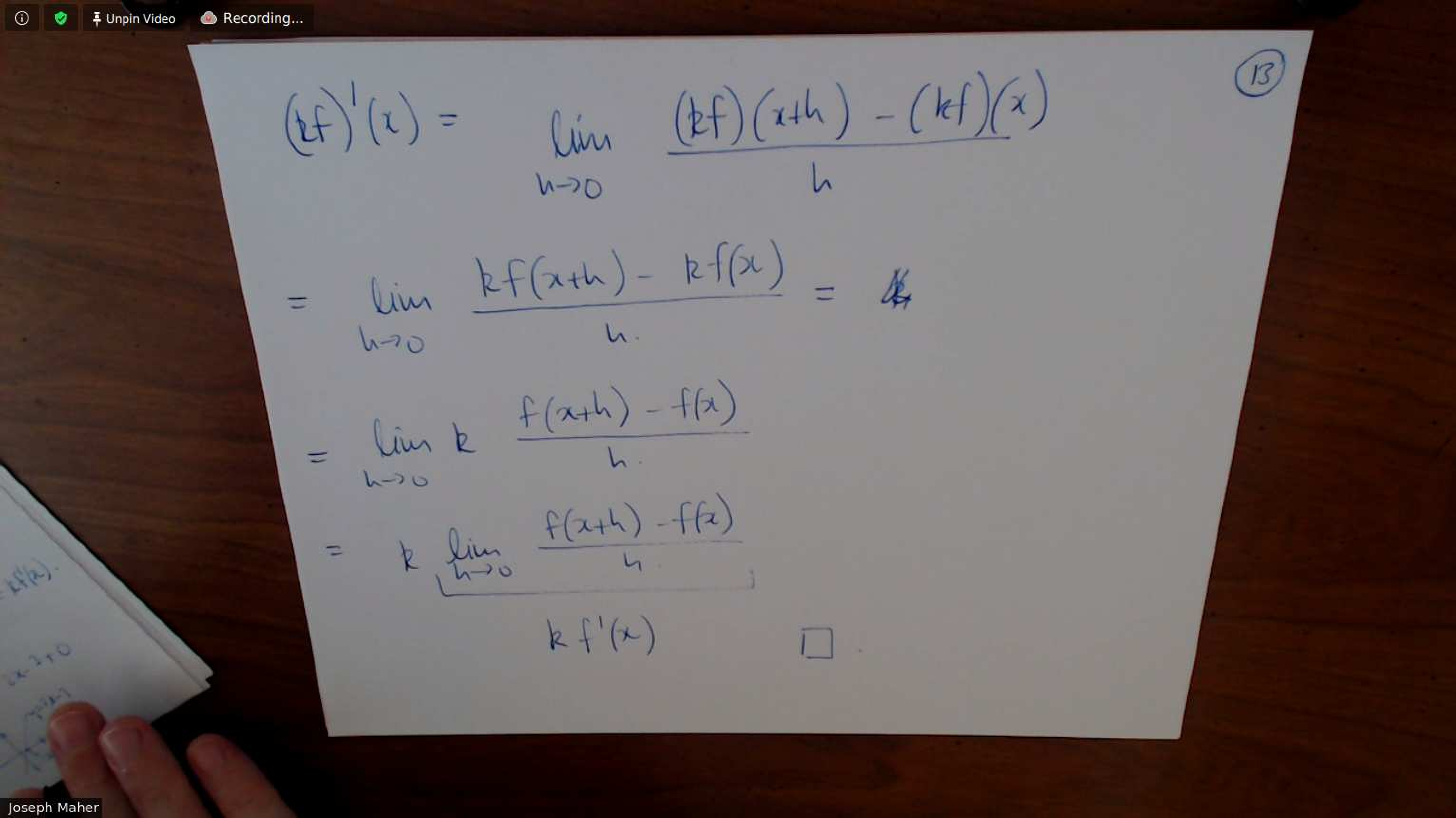
$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$f'(x) \qquad + \qquad g'(x)$$



⑬

$$(kf)'(x) = \lim_{h\to 0} \frac{(kf)(x+h) - (kf)(x)}{h}$$

$$= \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} = \cancel{k}$$

$$= \lim_{h\to 0} k \frac{f(x+h) - f(x)}{h}$$

$$= k \underbrace{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}_{kf'(x)}$$

$$\square$$

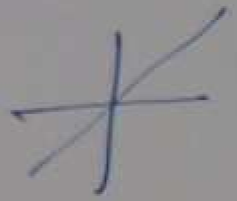
Example $f(x) = x^2 - 3x + 2 = (x-1)(x-2).$ $(x^n)' = nx^{n-1}$

find $f'(x) = \frac{d}{dx}\left(x^2 - 3x + 2\right).$

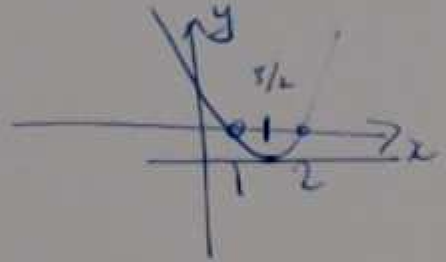
$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(-3x) + \frac{d}{dx}(2).$$

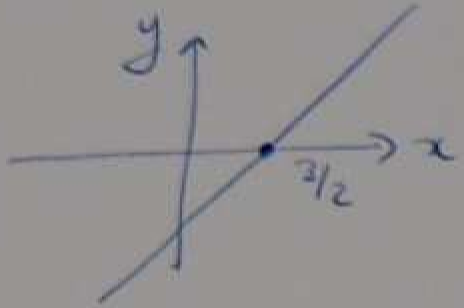
$$= 2x - 3\frac{d}{dx}(x) + 0$$

$$= 2x - 3.$$



graphs.





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(14)

Example $f(x) = x^2 - 3x + 2 = (x-1)(x-2)$.

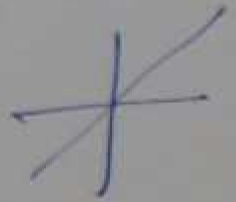
$(x^n)' = nx^{n-1}$

find $f'(x) = \frac{d}{dx}\left(x^2 - 3x + 2\right)$.

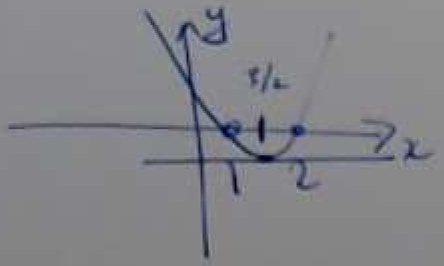
$= \frac{d}{dx}(x^2) + \frac{d}{dx}(-3x) + \frac{d}{dx}(2)$.

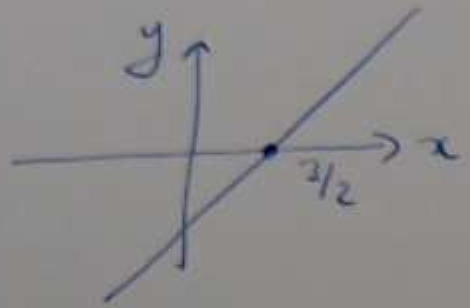
$= 2x - 3\frac{d}{dx}(x) + 0$.

$= 2x - 3$.



graphs





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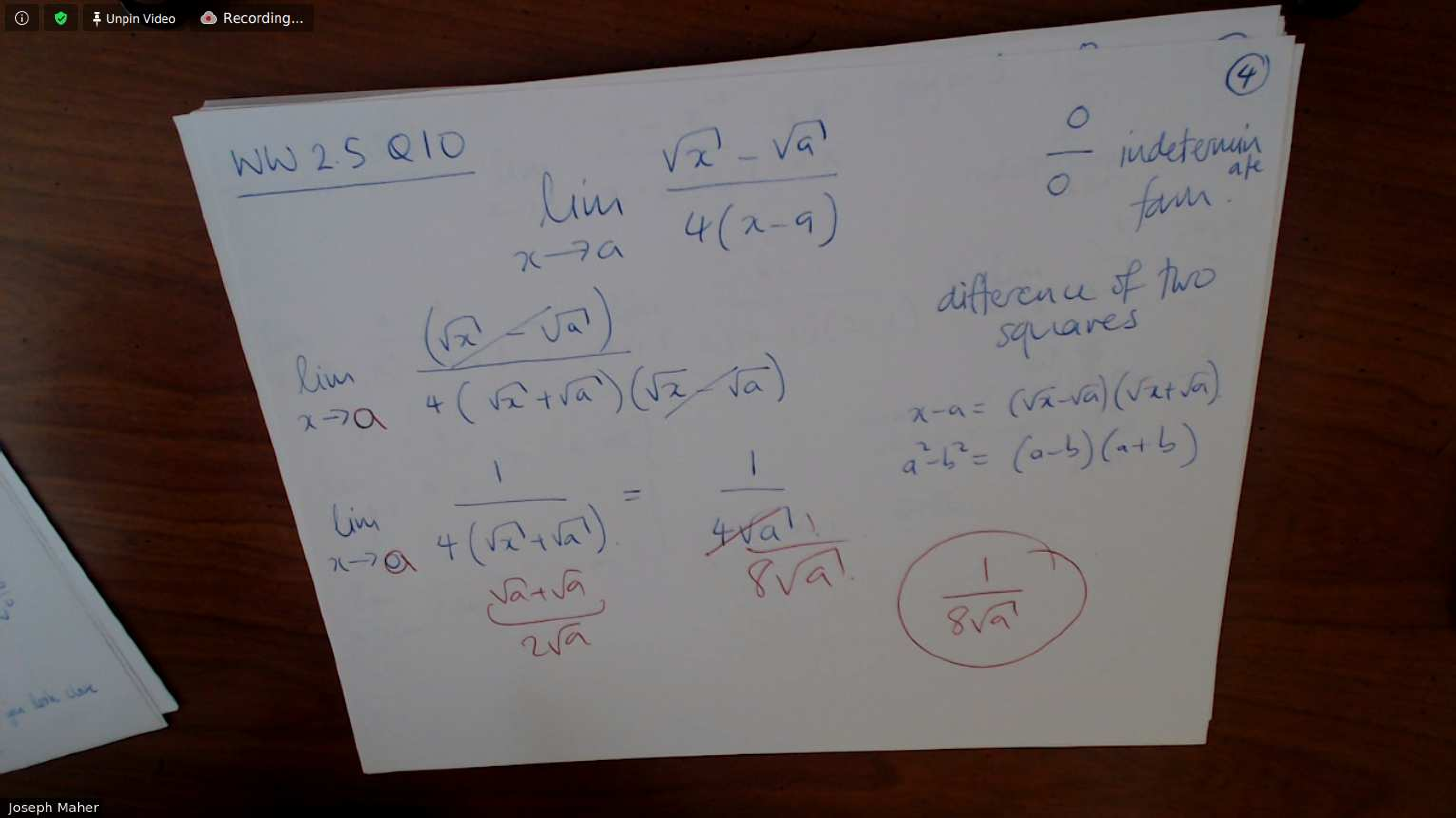


④

WW 2.5 Q10

$$\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{4(x-a)}$$

$\frac{0}{0}$ indeterminate form.

$$\lim_{x\to a} \frac{\cancel{(\sqrt{x}-\sqrt{a})}}{4(\sqrt{x}+\sqrt{a})\cancel{(\sqrt{x}-\sqrt{a})}}$$

difference of two squares

$x-a = (\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})$

$a^2-b^2 = (a-b)(a+b)$

$$\lim_{x\to a} \frac{1}{4\underbrace{(\sqrt{x}+\sqrt{a})}_{\substack{\sqrt{a}+\sqrt{a}\\ 2\sqrt{a}}}} = \frac{1}{\cancel{4\sqrt{a}}\;8\sqrt{a}}$$

$$\boxed{\frac{1}{8\sqrt{a}}}$$ ✓

Fact: we can differentiate all polynomials.

$$\frac{d}{dx}\left(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0\right) = n a_n x^{n-1} + a_{n-1}(n-1)x^{n-2} + \cdots + a_1 + 0.$$

Derivative of e^x

consider $f(x) = b^x$ $\quad b > 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}.$$

$$= \lim_{h \to 0} \frac{b^x \cdot b^h - b^x}{h} = \lim_{h \to 0} b^x \cdot \frac{b^h - 1}{h}.$$

Fact: we can differentiate all polynomials.

$$\frac{d}{dx}\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0\right) = n a_n x^{n-1} + a_{n-1}(n-1)x^{n-2} + \cdots + a_1 + 0.$$

Derivative of e^x

consider $f(x) = b^x$ $\quad b > 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}.$$

$$= \lim_{h \to 0} \frac{b^x \cdot b^h - b^x}{h} = \lim_{h \to 0} b^x \cdot \frac{b^h - 1}{h}.$$

$$\frac{d}{dx}(b^x) = b^x \cdot \underbrace{\lim_{h\to 0} \frac{b^h - 1}{h}}_{\text{doesn't depend on } x!} = m_b \cdot b^x.$$

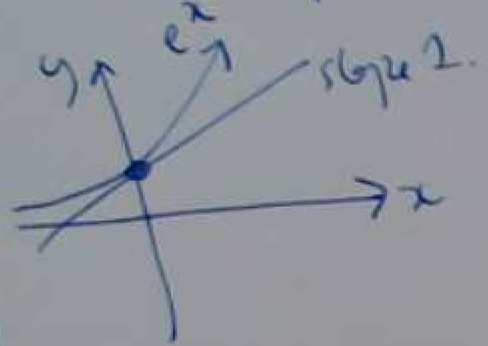
assume this limit exists,
call it m_b.

we've shown: the slope of the exponential function is proportional to its value.

$$\frac{d}{dx}(b^x) = m_b \cdot b^x.$$

in particular, slope at $x=0$ $\quad f'(0) = m_b b^0 = m_b$.

Recall: e is the special number such that the slope of e^x at $x=0$ is equal to 1.



$$\boxed{\begin{array}{l} f(x) = e^x, \quad f'(x) = e^x. \\ \frac{d}{dx}(e^x) = e^x. \end{array}}$$

Example $\frac{d}{dx}(7e^x + 4x^2)$. $(x^n)' = nx^{n-1}$

$$= \frac{d}{dx}(7e^x) + \frac{d}{dx}(4x^2).$$

$$= 7\frac{d}{dx}(e^x) + 4\frac{d}{dx}(x^2) = 7e^x + 8x.$$

observation e^x is not a polynomial.

$\frac{d}{dx}(e^x) = e^x$.

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f^{(3)}(x) = e^x$

$\vdots$

↑ never differentiates to 0

$\frac{d}{dx}(x^2)$

$f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2$

$f^{(3)}(x) = 0$.

$f(x) = x^{100}$

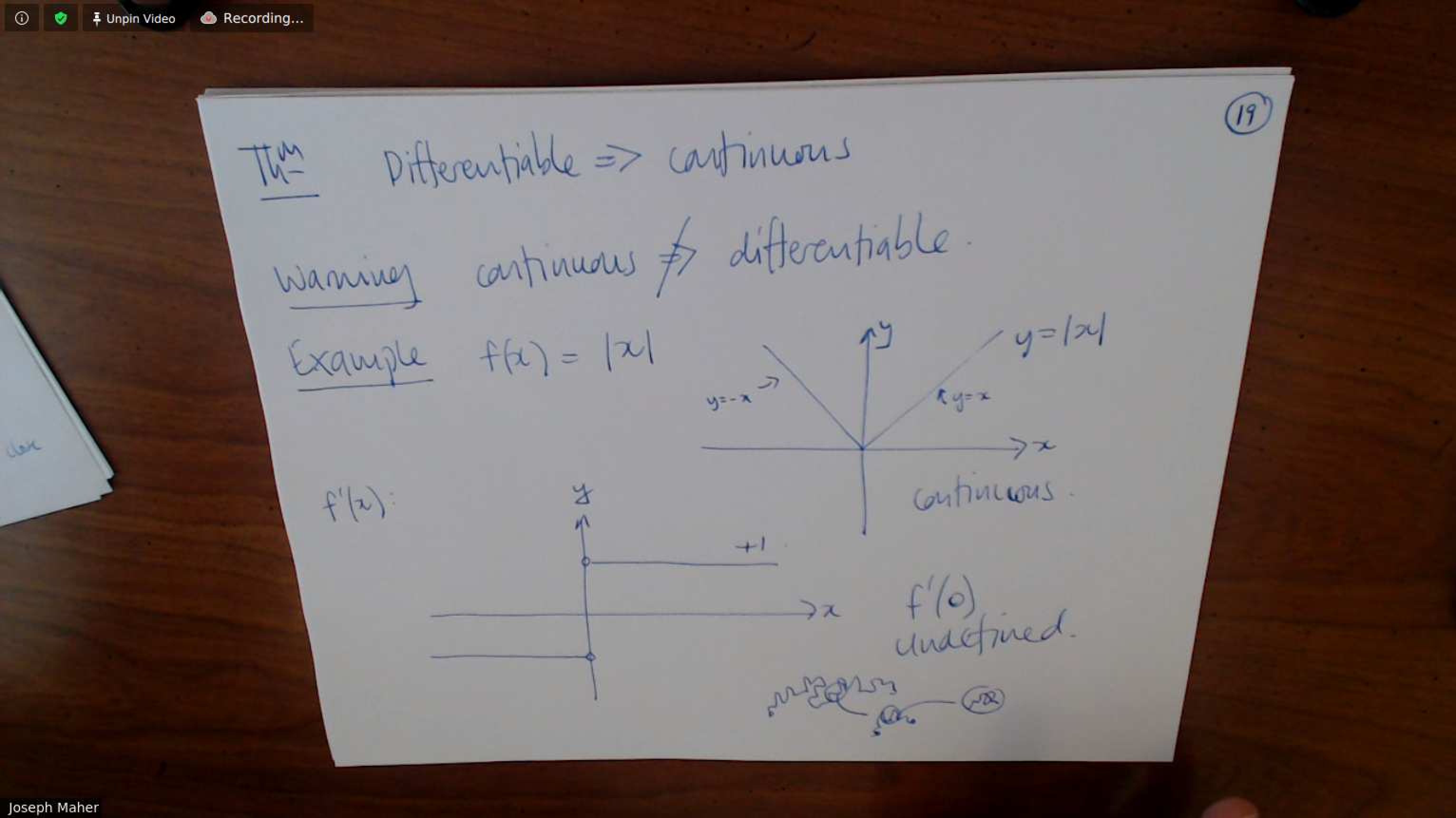
$f'(x) = 100x^{99}$

$f''(x)\ 100 \times 99 x^{98}$

$\vdots$

0.

↑ any polynomial ~~eve~~ eventually differentiates to 0.



Thm Differentiable $\Rightarrow$ continuous

Warning continuous $\not\Rightarrow$ differentiable.

Example $f(x) = |x|$

continuous.

$f'(x)$:

$f'(0)$ undefined.

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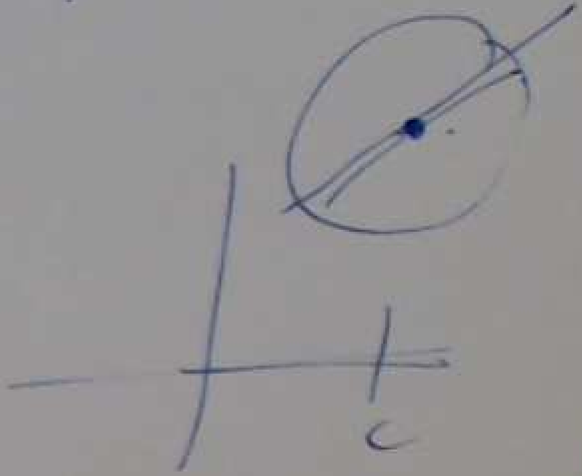
local picture if $f(x)$ is differentiable at $x=c$

then the graph close to c looks like a straight line



§ 3.3 Product and quotient rules

new functions from old

$f(x)g(x)$ product

$\frac{f(x)}{g(x)}$ quotient

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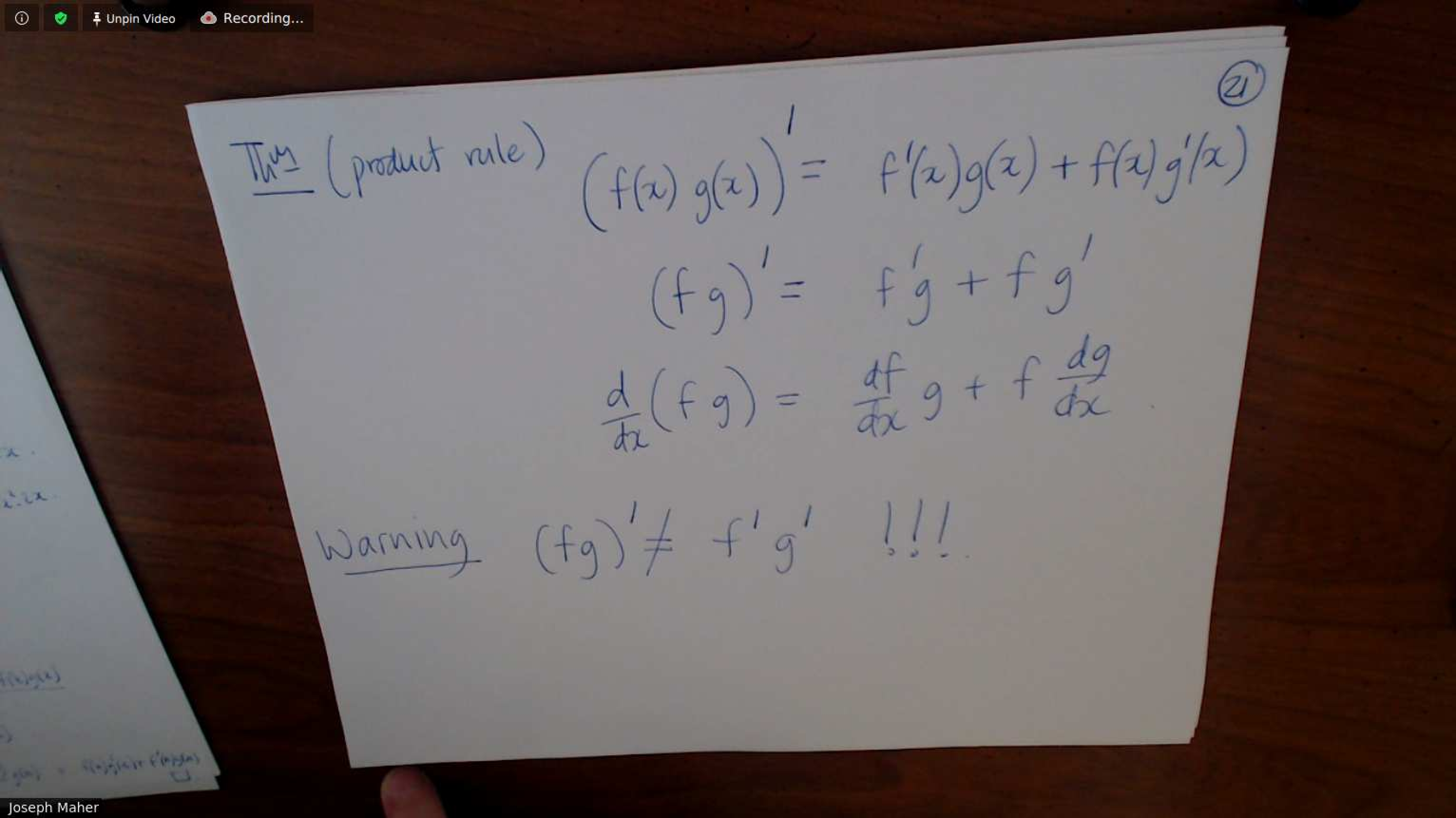
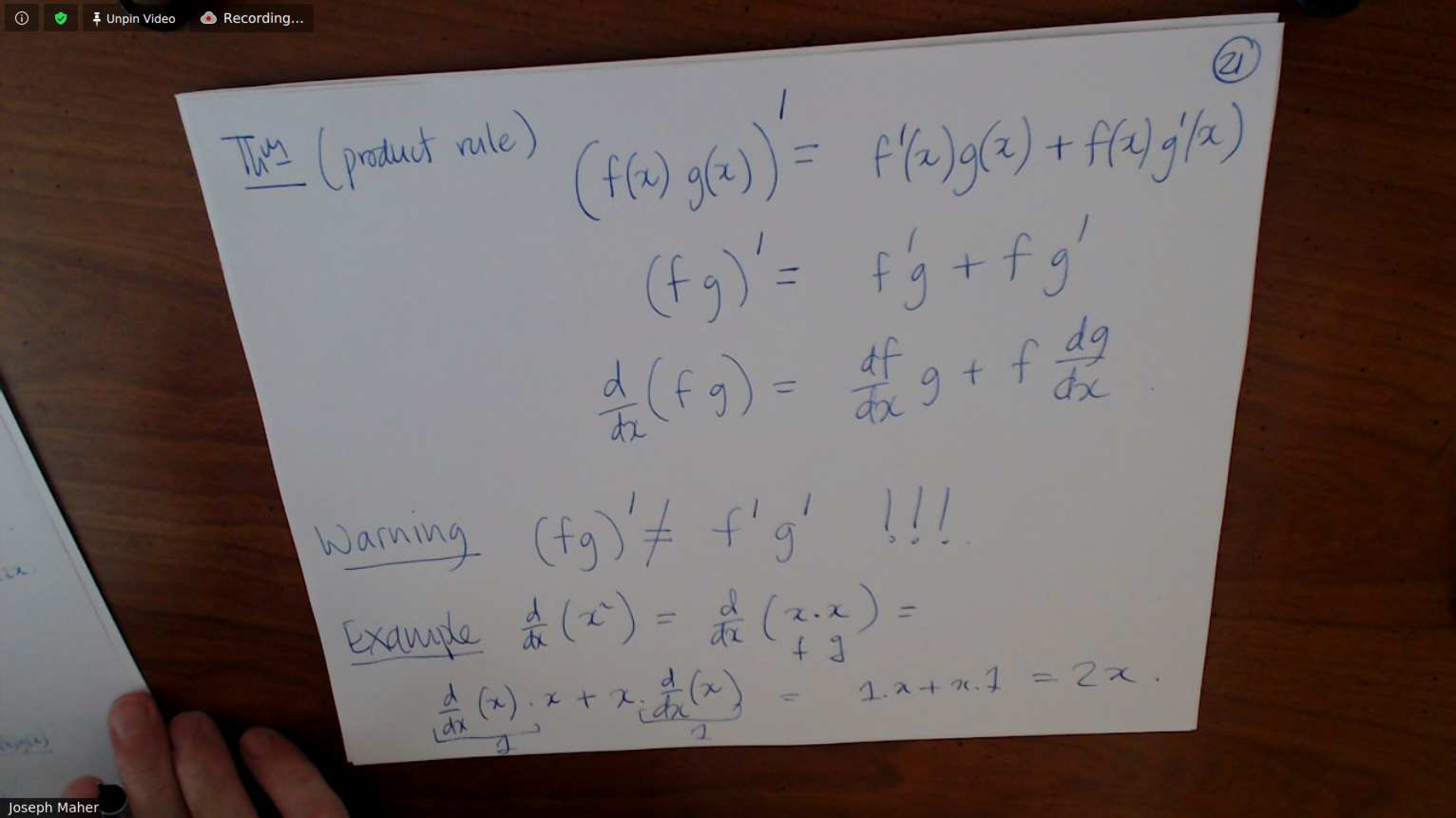


Thm (product rule) $(f(x)\,g(x))' = f'(x)g(x) + f(x)g'(x)$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Warning $(fg)' \neq f'g'$!!!



Thm (product rule) $(f(x)\,g(x))' = f'(x)g(x) + f(x)g'(x)$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}.$$

Warning $(fg)' \neq f'g'$!!!

Example $\frac{d}{dx}(x^2) = \frac{d}{dx}(\underset{f}{x}\cdot\underset{g}{x}) =$

$$\underbrace{\frac{d}{dx}(x)}_{1}\cdot x + x\cdot\underbrace{\frac{d}{dx}(x)}_{1} = 1\cdot x + x\cdot 1 = 2x.$$

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② $$\frac{d}{dx}\left(\underbrace{3x^2}_{f}\underbrace{(x^2+1)}_{g}\right)=\left[\frac{d}{dx}(3x^2)\right](x^2+1)+3x^2\frac{d}{dx}(x^2+1).$$

$$= 6x(x^2+1)+3x^2(2x+0).$$

$$= 6x(x^2+1)+6x^3.$$

③ $$\frac{d}{dx}\left(\underset{f}{x^2}\,\underset{g}{e^x}\right) = \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x).$$

$$2xe^x + x^2e^x.$$

$$= 2xe^x + x^2e^x.$$

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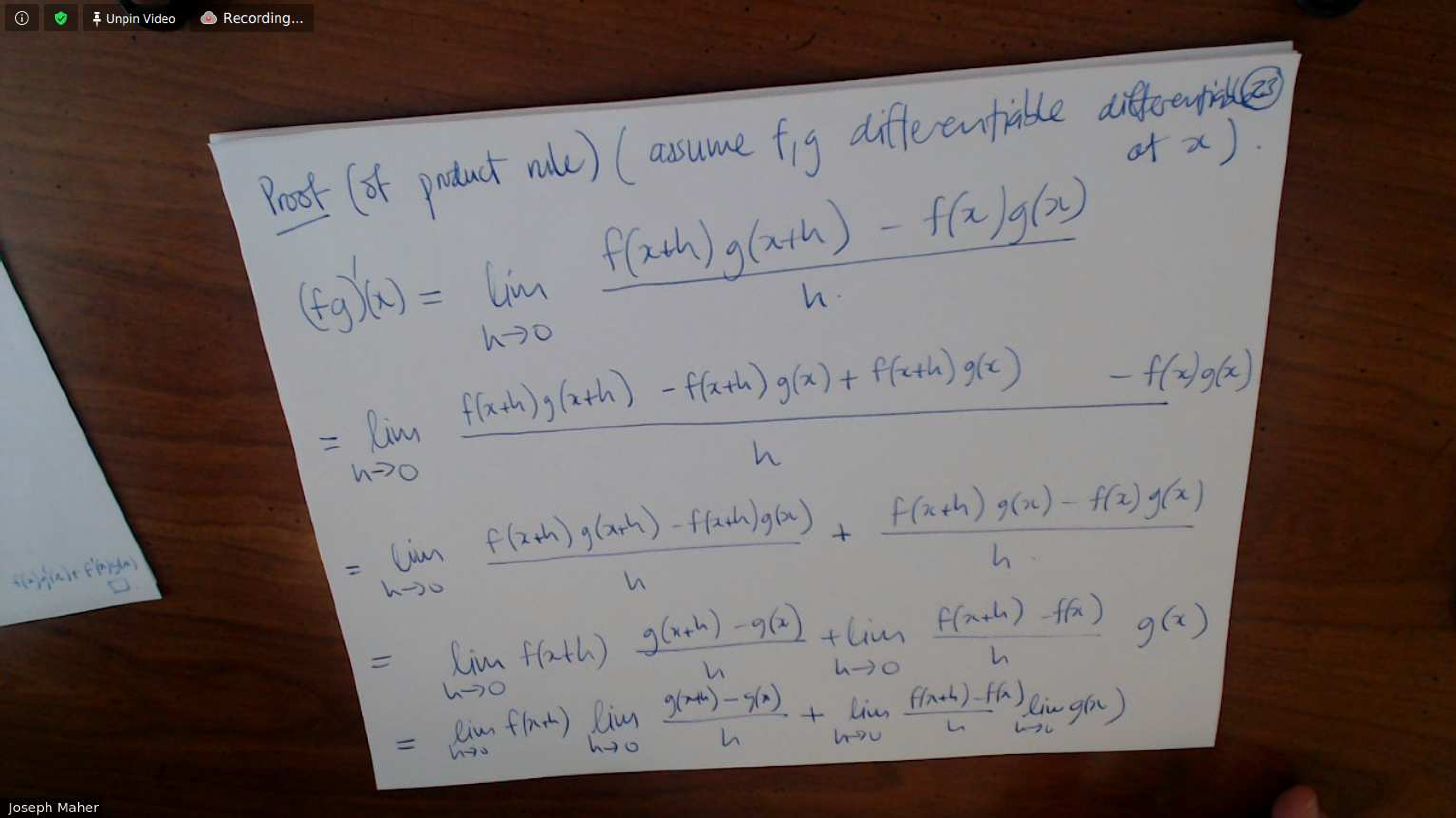

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<u>Proof</u> (of product rule) (assume f, g differentiable differentiable at x).

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h}.$$

$$= \lim_{h\to 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x)$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x))$$

$$f(x)\,g'(x) + f'(x)\,g(x) \quad \square .$$

<u>summary</u>

$$(fg)' = f'g + fg' .$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx} .$$