WW 2.6 Q3

$$\lim_{x \to 0} \frac{x^2}{\sin^2(5x)}$$

use:

- limit rules.
- $\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$

$$\lim_{x \to 0} \frac{x}{\sin(5x)} \cdot \frac{x}{\sin(5x)} = \underbrace{\lim_{x \to 0} \frac{x}{\sin(5x)}}\cdot \underbrace{\lim_{x \to 0} \frac{x}{\sin(5x)}}.$$

$$\lim_{x \to 0} \frac{x}{\sin(5x)} = \frac{1}{\lim_{x \to 0} \frac{\sin(5x)}{x}} = \frac{1}{5}$$

$5x = \theta \Leftrightarrow x = \theta/5$

$$\lim_{x \to 0} \frac{\sin(\theta)}{\theta/5}.$$

$$= \lim_{x \to 0} 5 \frac{\sin\theta}{\theta}.$$

$$= 5 \boxed{\lim_{x \to \theta} \frac{\sin\theta}{\theta}} = 1.$$

$$= 5.$$

$$\lim_{x \to 0} \frac{x^2}{\sin^2(5x)} = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}.$$

WW 26 Q3.

$$\lim_{x\to 0} \frac{x^2}{\sin^2(5x)} = \lim_{x\to 0} \left(\frac{x}{\sin(5x)}\right)^2.$$

recall $f(x)$ is cts at $\lim_{x\to 0} g(x) = L$

then $\lim_{x\to 0} f(g(x)) = f\left(\lim_{x\to 0} g(x)\right)$.

$$\lim_{x\to 0} \frac{x}{\sin(5x)} \overset{\frac{\sin\theta}{\theta}}{\rightsquigarrow} \frac{1}{5}. \qquad \lim_{x\to 0} = \left(\frac{1}{5}\right)^2 = \frac{1}{25}.$$

WW 2.6 Q10

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

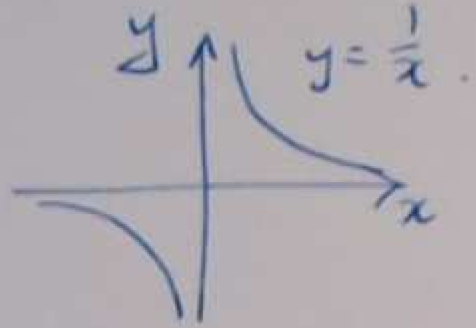
$$\lim_{\theta \to 0} \frac{4\sin(2\theta) - 8\sin\theta}{\theta^2}$$

$$\lim_{\theta \to 0} \frac{8\sin\theta\cos\theta - 8\sin\theta}{\theta^2} = 8\lim_{\theta \to 0} \frac{\sin\theta\cos\theta - \cancel{8}\sin\theta}{\theta^2}.$$

$$= 8\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\frac{(\cos\theta - 1)}{\theta} = 8\underbrace{\left(\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\right)}_{1}\underbrace{\left(\lim_{\theta \to 0}\frac{\cos\theta - 1}{\theta}\right)}_{0}.$$

$$= 0.$$

§2.7 Limits at infinity



3 $$\lim_{x\to\infty} \frac{1}{x} = 0.$$

$$\lim_{x\to\infty} \frac{1}{x} - \frac{2}{3x+1} = \underbrace{\lim_{x\to\infty} \frac{1}{x}}_{0} - \underbrace{\lim_{x\to\infty} \frac{2}{3x+1}}_{0}$$

$$\lim_{x\to\infty} \frac{2/x}{3+1/x}$$

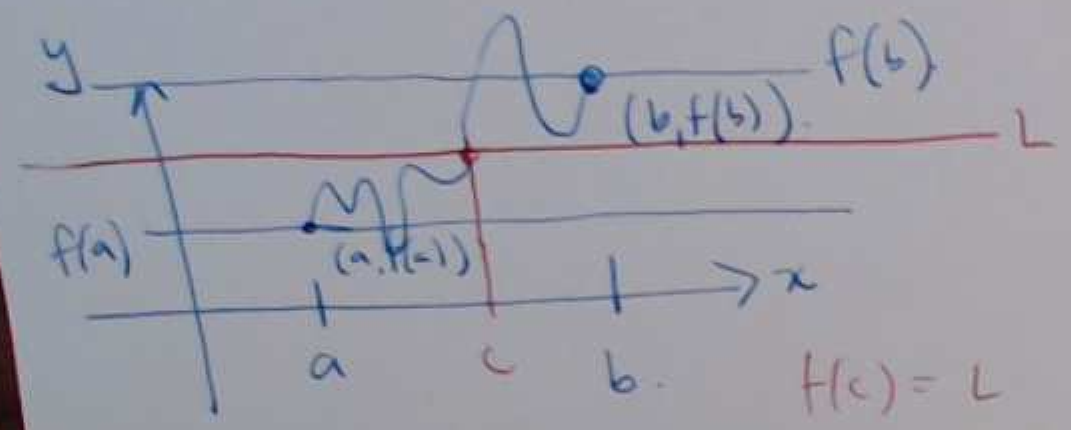
$$= 0 - 0 = 0$$

4 $$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{4x+1} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{4+1/x} = \frac{\sqrt{2}}{4}.$$

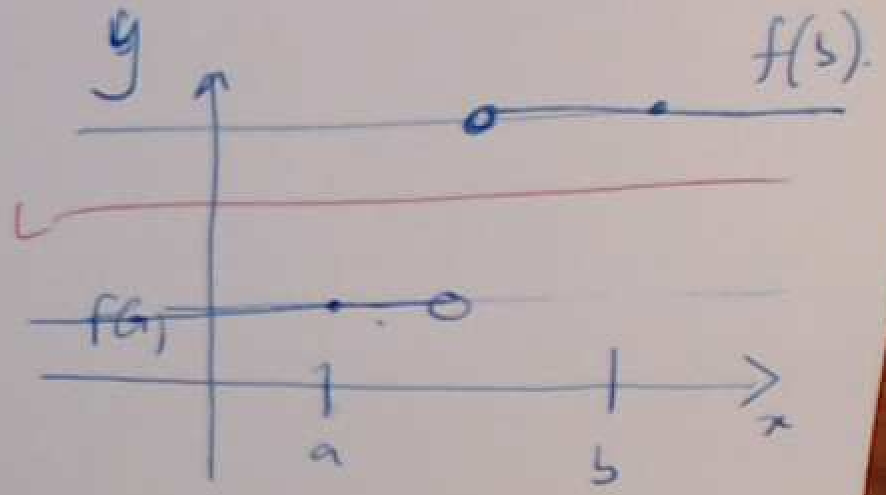
$$\frac{\sqrt{2x^2+1}}{x} = \frac{\sqrt{2x^2+1}}{\sqrt{x^2}} = \sqrt{\frac{2x^2+1}{x^2}}.$$

§2.8 Intermediate value theorem (IVT)

"continuous functions can't skip values".



f continuous.

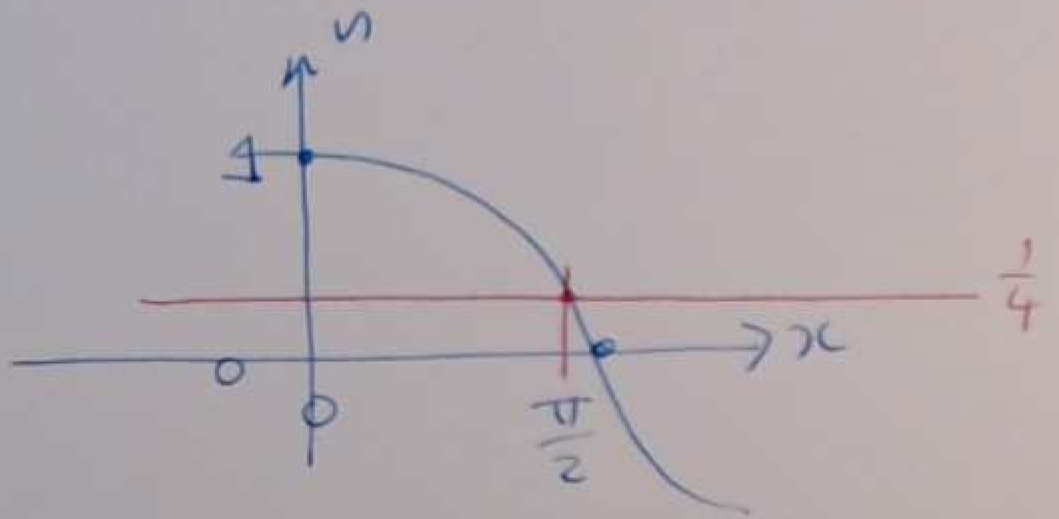


f not continuous

Thm (IVT) If $f(x)$ is cts on $[a,b]$ and $f(a) \neq f(b)$ then for any L between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ s.t. $f(c) = L$. $\square$

Example show $\cos(x) = \frac{1}{4}$ has at least one solution.

consider $[0, \frac{\pi}{2}]$.



$\cos(0) = 1$

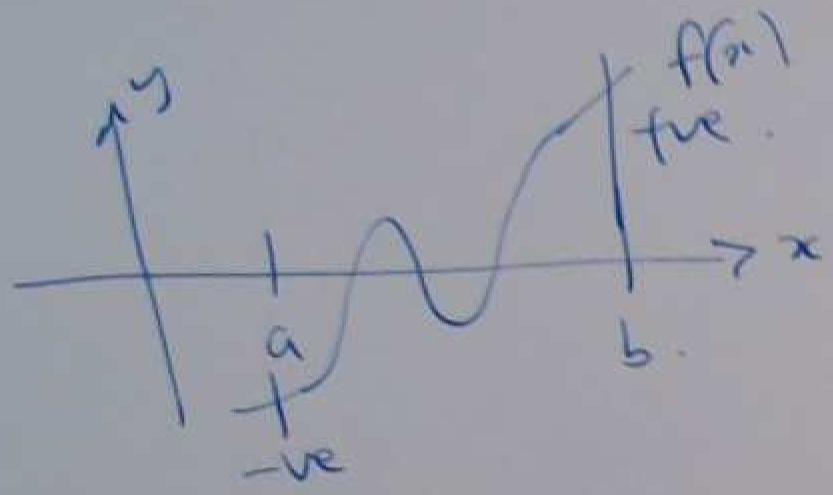
$\cos\left(\frac{\pi}{2}\right) = 0$

$0 \leq \frac{1}{4} \leq 1$ IVT $\Rightarrow \exists\, c \in [0, \frac{\pi}{2}]$ s.t. $f(c) = \cos(c) = \frac{1}{4}$.

special case: finding zeros.

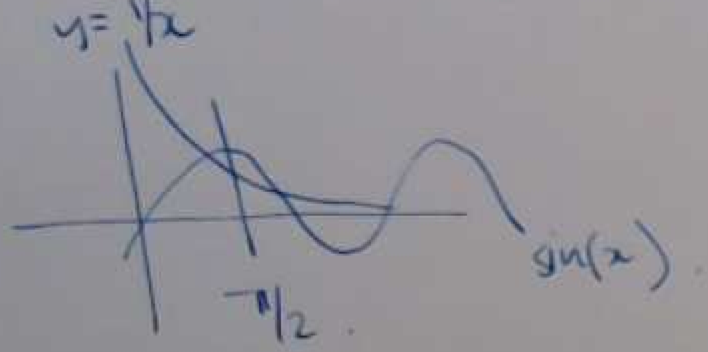
Corollary if $f(x)$ is cts $[a,b]$ and $f(a), f(b)$ have different signs, then $\exists\, c \in [a,b]$ s.t. $f(c) = 0$.

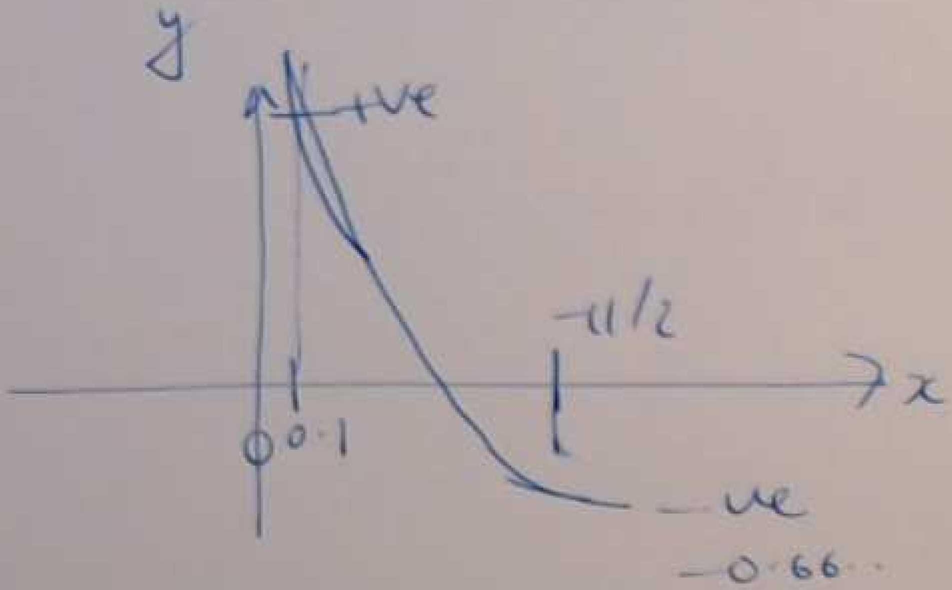
bisection method



example $\sin(x) = \frac{1}{x}$

consider $f(x) = \frac{1}{x} - \sin(x)$

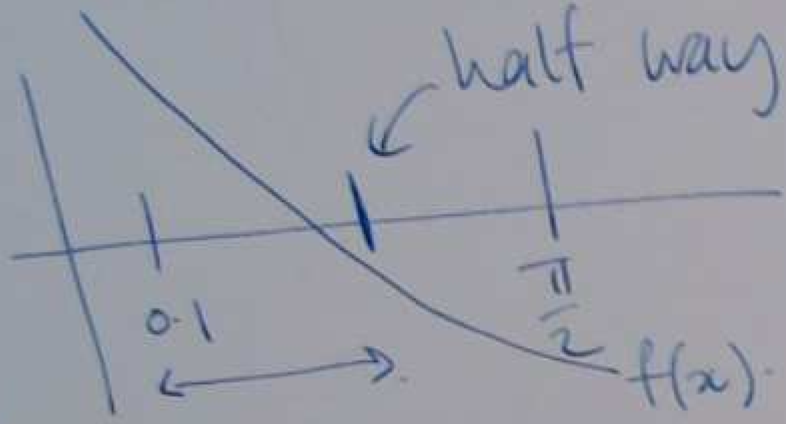




$\sin(\frac{\pi}{2}) = 1$

$\frac{1}{\pi/2} \approx \frac{1}{1.5} \approx 0.66$

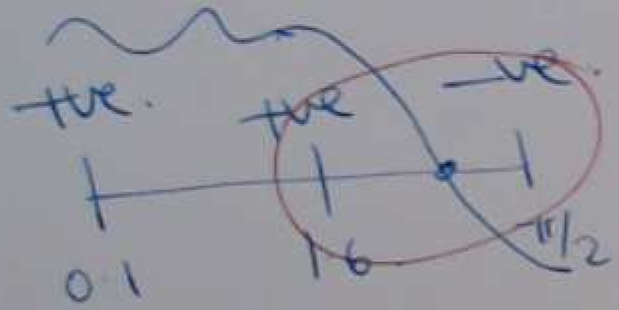
IVT $\Rightarrow$ there is a solution between 0.1 and $\frac{\pi}{2}$.



$$\frac{0.1+\frac{\pi}{2}}{2} \approx 1.6\ldots$$

evaluate $f(1.6)$

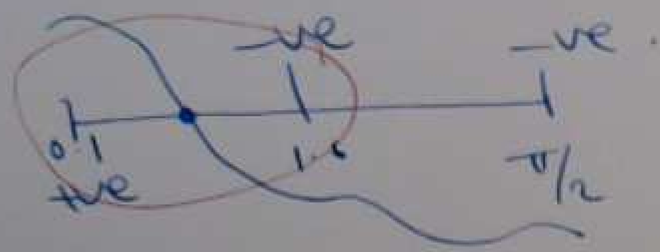
+ve or −ve.



at exactly one of

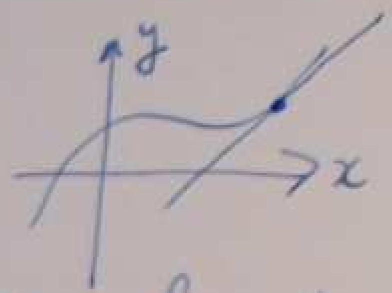
$[0.1, 1.6]$ $[1.6, \pi/2]$.

has the property that ~~has the~~ function has different signs at each end point

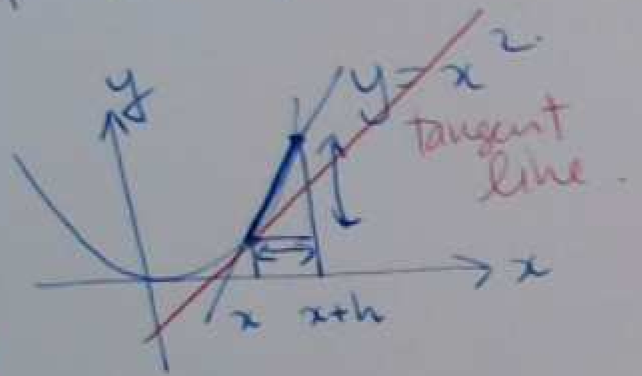


keep on subdividing.

§3.1 Defn of the derivative



recall: we can compute the average rate of change of a function over an interval

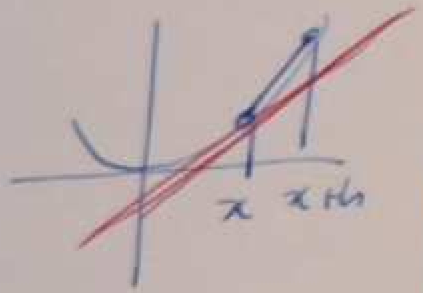


$[x, x+h]$: $\dfrac{f(x+h)-f(x)}{x+h-x}$.

Q: how do we find the slope of the tangent line?

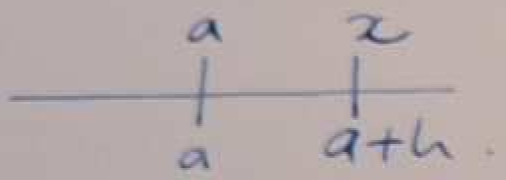
idea: look at the average rate of change over $[x, x+h]$ and take limit as $h \to 0$.

Def: the slope of the tangent line to $f(x)$ at $x = a$ is $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

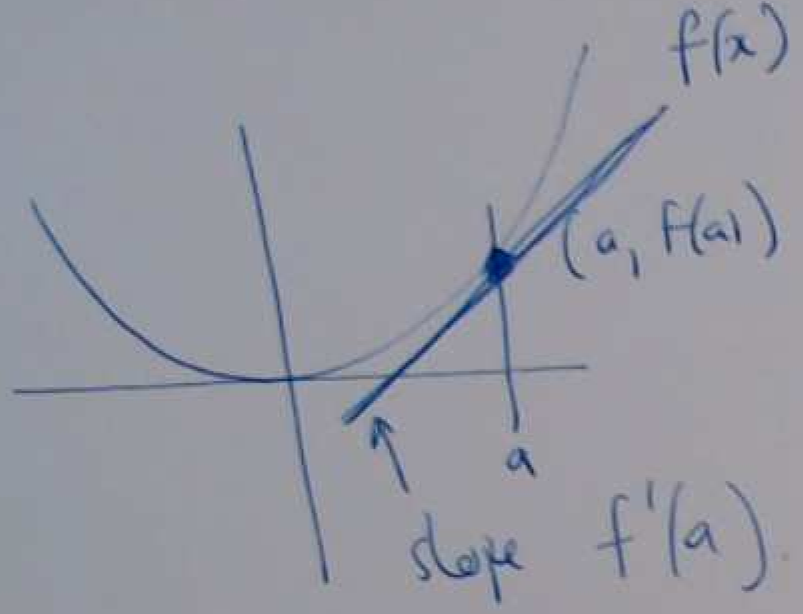


notation also called the **derivative**

written $\underbrace{f'(a)}_{\text{Newton}}$ $\underbrace{\frac{df}{dx}(a)}_{\text{Leibnitz}}$.



note $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ same as $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$



tangent line.

$$y - f(a) = f'(a)(x-a).$$

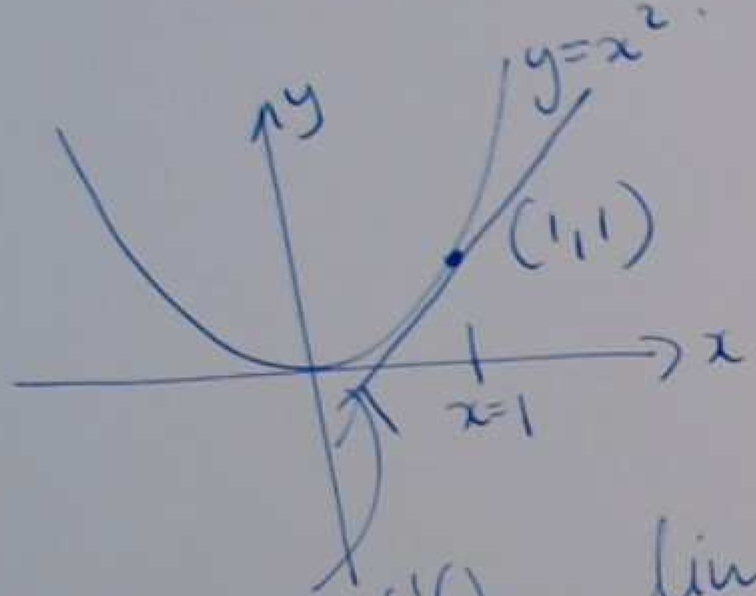
$$y - y_0 = m(x - x_0).$$

<u>Defn</u> the tangent line to $f(x)$ at the point $(a, f(a))$ is the straight line through that point with slope $f'(a)$.

$$y - f(a) = f'(a)(x-a) \leftrightarrow \quad y = f(a) + f'(a)(x-a)$$

<u>Example</u> find the tangent line to $f(x) = x^2$ at $x = 1$.

slope $f'(1)$



slope $f'(1)$
$= 2$.

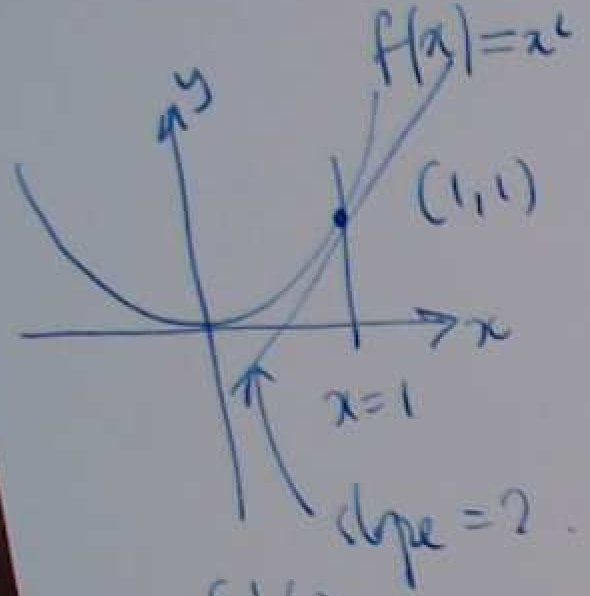
$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$$

$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{h}.$$

$$\lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} = \lim_{h \to 0} \frac{2h + h^2}{h}.$$

$$= \lim_{h \to 0} 2 + h = 2.$$

tangent line.



$f'(1)=2$.

$$y - f(a) = f'(a)(x-a).$$

$$y-1 = 2(x-1).$$

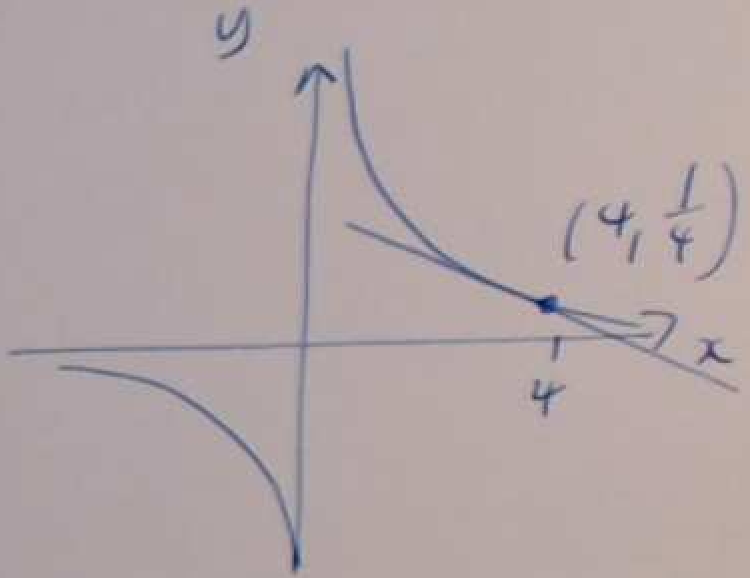
$$y-1 = 2x-2$$

$$y = 2x-1.$$

equation for tangent line.

Example find slope of tangent line to

$$f(x) = \frac{1}{x} \quad \text{at } x = 4.$$



$$f'(4) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

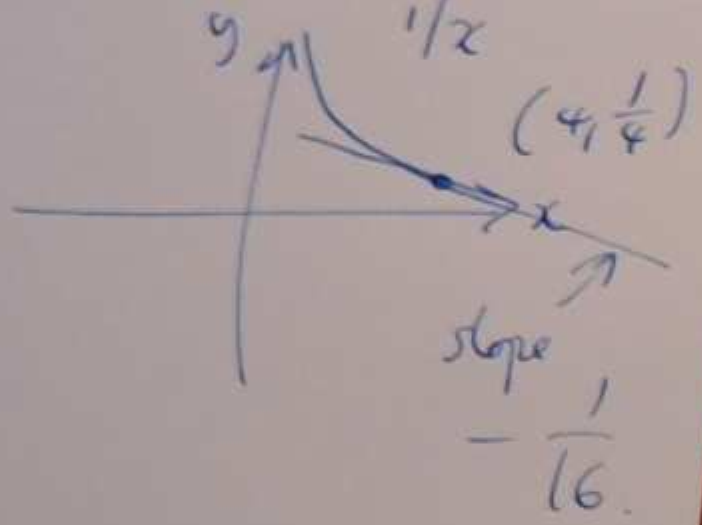
$$= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4 \times 1}{4 \times (4+h)} - \frac{1 \times (4+h)}{(4) \times (4+h)}}{h} = \lim_{h \to 0} \frac{\frac{4 - (4+h)}{(4+h)\,4}}{h}$$

$$\lim_{h\to 0} \frac{4-(4+h)}{(4+h)4h} = \lim_{h\to 0} \frac{\cancel{4}-\cancel{4}-h}{(4+h)4h}$$

$$= \lim_{h\to 0} \frac{-h}{(4+h)4h} = \lim_{h\to 0} \frac{-1}{(4+h)4} = -\frac{1}{16}.$$

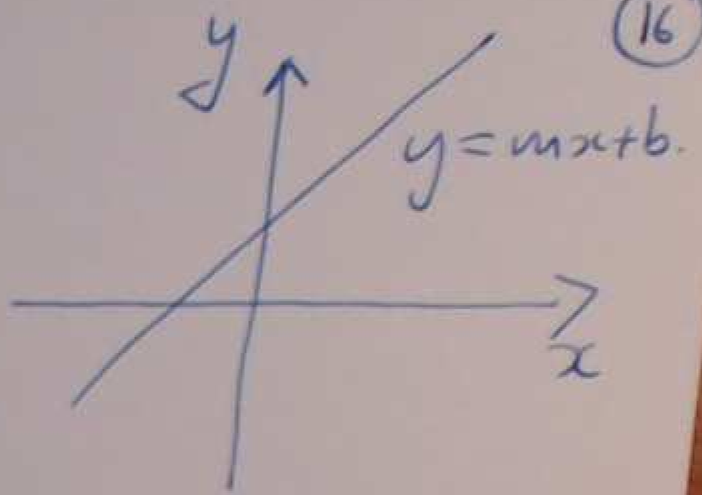
$$f'(4) = -\frac{1}{16}$$



equation of tangent line

$$y - \frac{1}{4} = -\frac{1}{16}(x-1)$$

<u>Example</u> straight line $y = mx + b$.



$f(x) = mx + b$.

find slope at $x = a$. $f'(a)$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{m(a+h) + b - (ma + b)}{h}$$

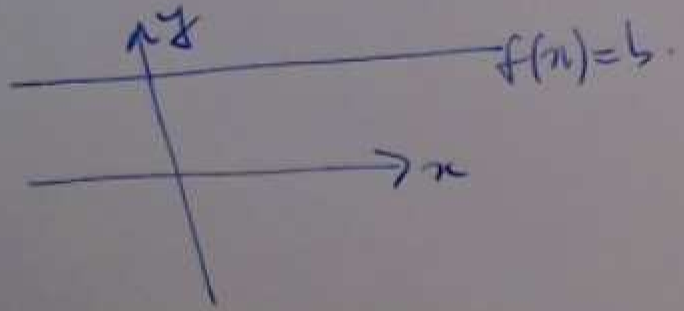
$$= \lim_{h \to 0} \frac{\cancel{ma} + mh + \cancel{b} - \cancel{ma} - \cancel{b}}{h}$$

$$= \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m.$$

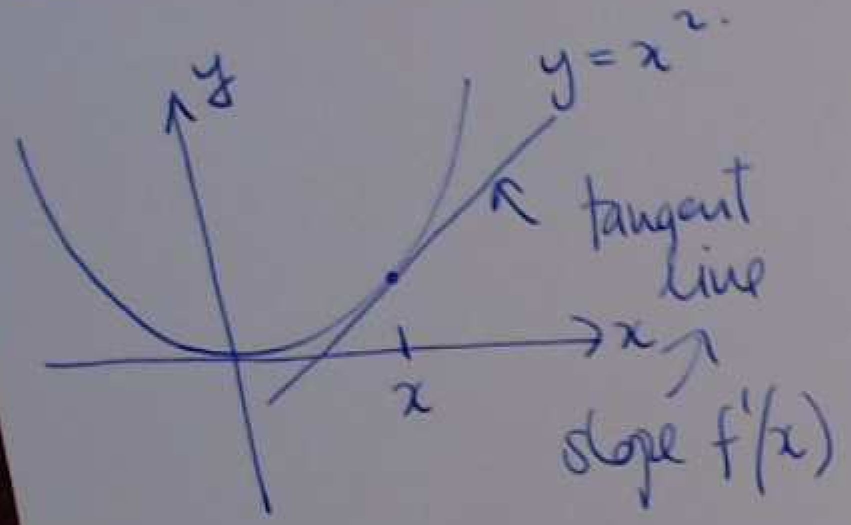
slope of straight line $y = mx + b$

at $x = a$ is $f'(a) = m$.

<u>observation</u> if $f(x) = b$ then $f'(x) = 0$
for all x



§3.2 Derivative as a function



$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x^2$$

at each input x there is a unique output.

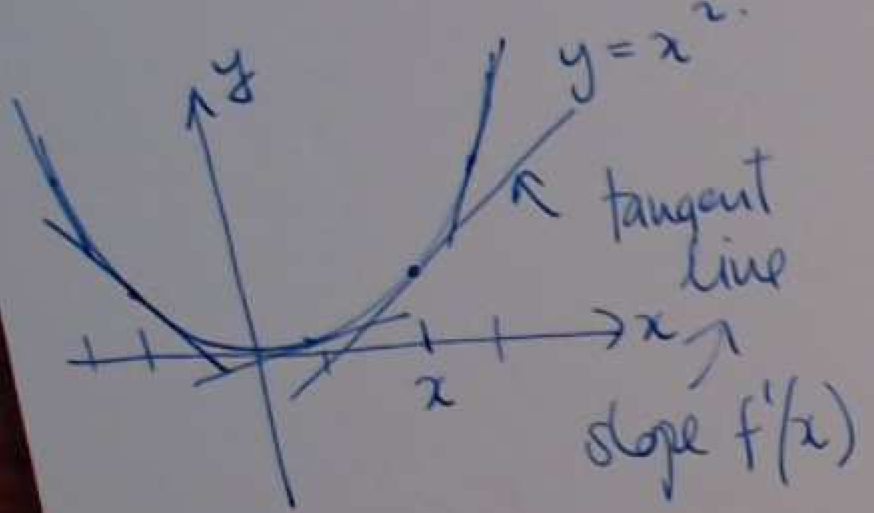
at each point x, there is a tangent line, which has a slope, which is a number

this defines a function $\quad f' : \mathbb{R} \longrightarrow \mathbb{R}$
$$x \longmapsto f'(x).$$

Recording Unpin Video

(18)

§3.2 Derivative as a function



$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x^2$$

at each input x there is a unique output.

at each point x, there is a tangent line, which has a slope, which is a number

this defines a function $f' : \mathbb{R} \longrightarrow \mathbb{R}$

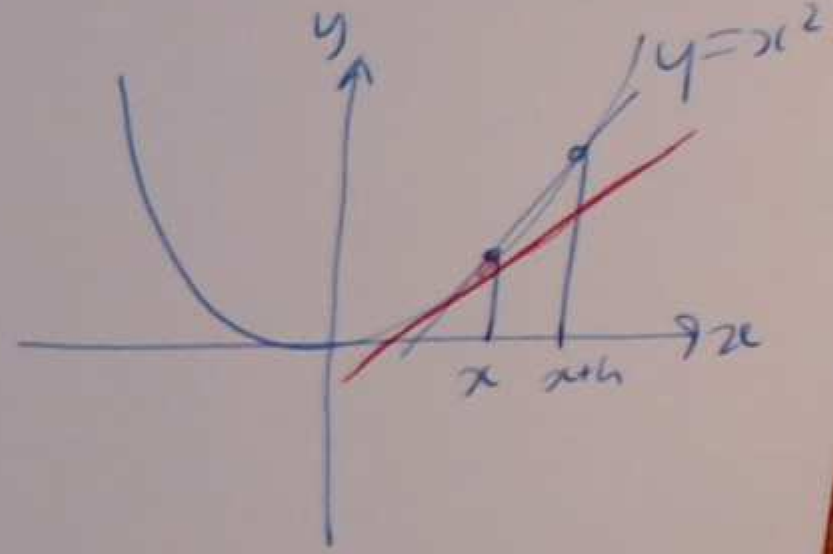
$$x \longmapsto f'(x).$$

Joseph Maher

$f(x) = x^2$

notation we call this function $f'(x)$

say "the derivative of f".



Example $f(x) = x^2$

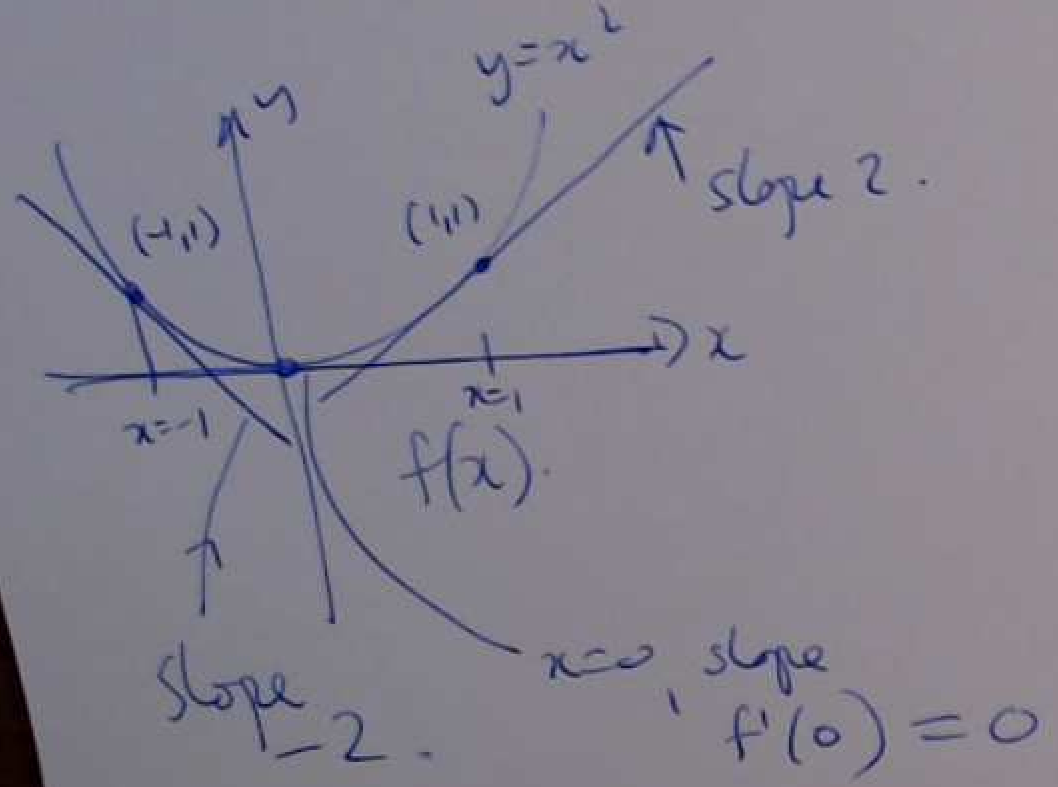
find slope of tangent line

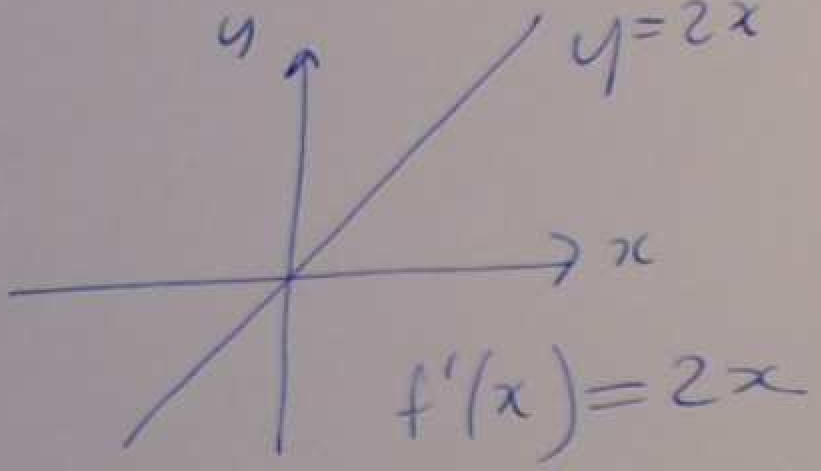
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

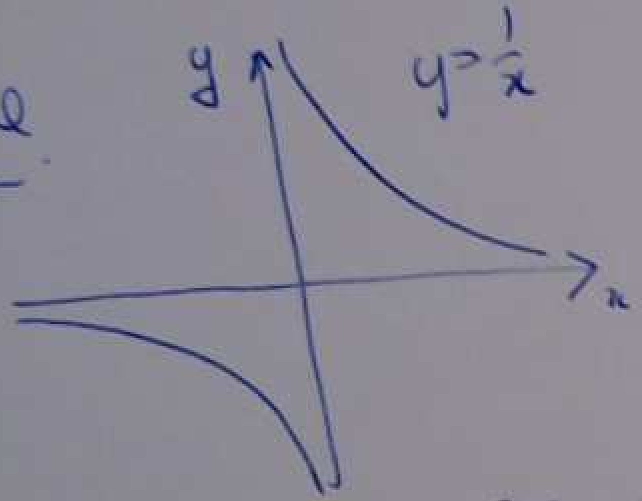
$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$$

summary if $f(x) = x^2$, then $f'(x) = 2x$.



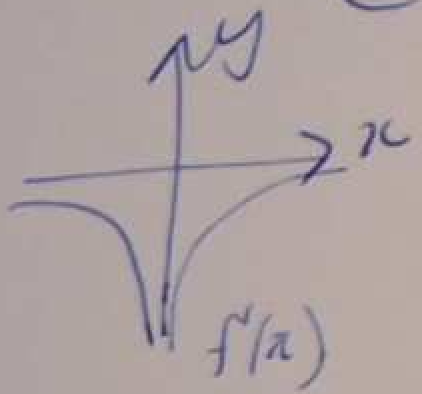


Example



$f(x) = \frac{1}{x}$.

find derivative.



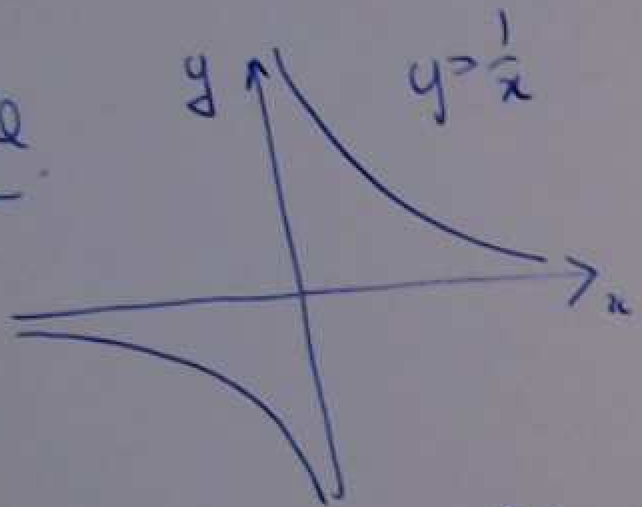
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}.$$

$$= \lim_{h\to 0} \frac{x - (x+h)}{(x+h)\,x\,h} = \lim_{h\to 0} \frac{\cancel{x} - \cancel{x} - h}{(x+h)\,x\,h}$$

$$= \lim_{h\to 0} \frac{-h}{(x+h)\,x\,h} = \lim_{h\to 0} \frac{-1}{(x+h)\,x} = \frac{-1}{x^2}.$$

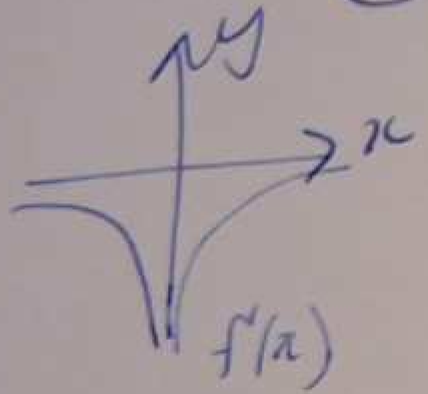
Conclusion $f'(x) = -\frac{1}{x^2}$.

Example



$f(x) = \frac{1}{x}$.

find derivative.



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}.$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{(x+h)\,x\,h} = \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - h}{(x+h)\,x\,h}$$

$$= \lim_{h \to 0} \frac{-h}{(x+h)\,x\,h} = \lim_{h \to 0} \frac{-1}{(x+h)\,x} = \frac{-1}{x^2}.$$

Conclusion $f'(x) = -\frac{1}{x^2}$.

Remarks

1. function $f: \text{domain} \longrightarrow \text{range}$.

ex: $f: \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto x^2$.

derivative: (differentiable) functions $\longrightarrow$ functions.

$$f(x) \longmapsto f'(x).$$

<u>warning</u>: not all functions are differentiable

2. "calculus" means rules for doing calculations, we won't have to compute limits all the time.

$$\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{4(x-a)}$$ plug in $x=a$. $\frac{0}{0}$. indeterminate

difference of two squares.

$$x-a = (\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a}).$$

$$\lim_{x\to a} \frac{\cancel{(\sqrt{x}-\sqrt{a})}}{4\cancel{(\sqrt{x}-\sqrt{a})}(\sqrt{x}+\sqrt{a})} = \lim_{x\to a} \frac{1}{4(\sqrt{x}+\sqrt{a})}.$$

$$= \cancel{\lim} \quad \frac{1}{4\cdot 2\sqrt{a}} = \frac{1}{8\sqrt{a}}$$