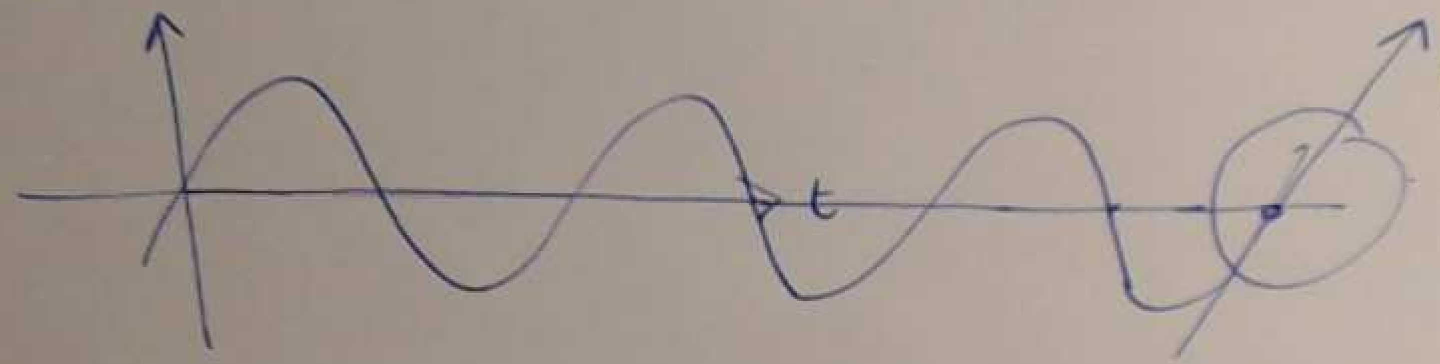
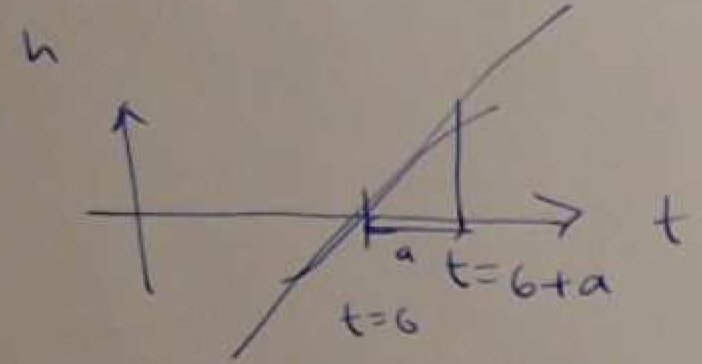
2.1 Q4.

$h(t) = 5\sin(2\pi t)$. velocity / speed $t = 6$. ①





approx rate of change

$$\frac{h(6+a)-h(6)}{a}.$$

$$\frac{5\sin\left(\boxed{\overset{12\pi+2\pi a}{2\pi(6+a)}}\right) - \overset{0}{5\sin(12\pi)}}{a}$$

$a = 0.1$

$a = 0.01$

...

0.000001

$\frac{5\sin(a)}{a}$ ✗

$$\frac{5\sin(\overset{2\pi a}{2\pi a})}{a}$$

$\simeq \cancel{5}$ 10π.

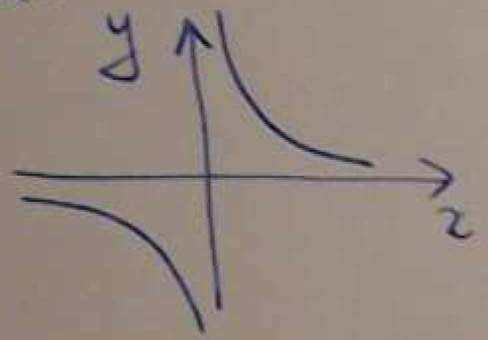
$31.4\ldots$

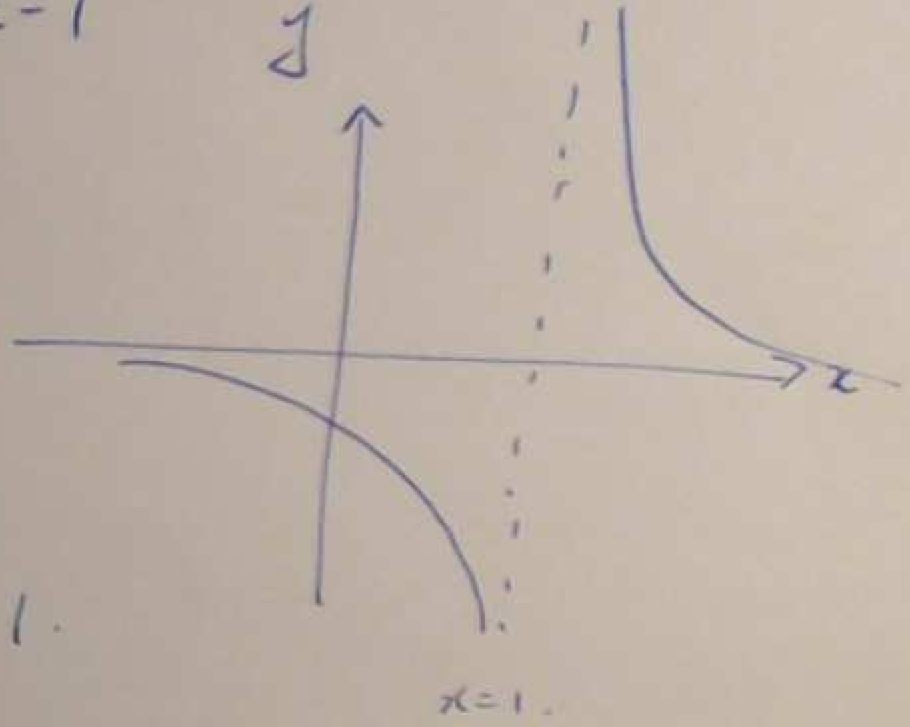
WW 2.4. Q1

$$f(x) = \frac{1}{x-1}$$

②

$y = \frac{1}{x}$.

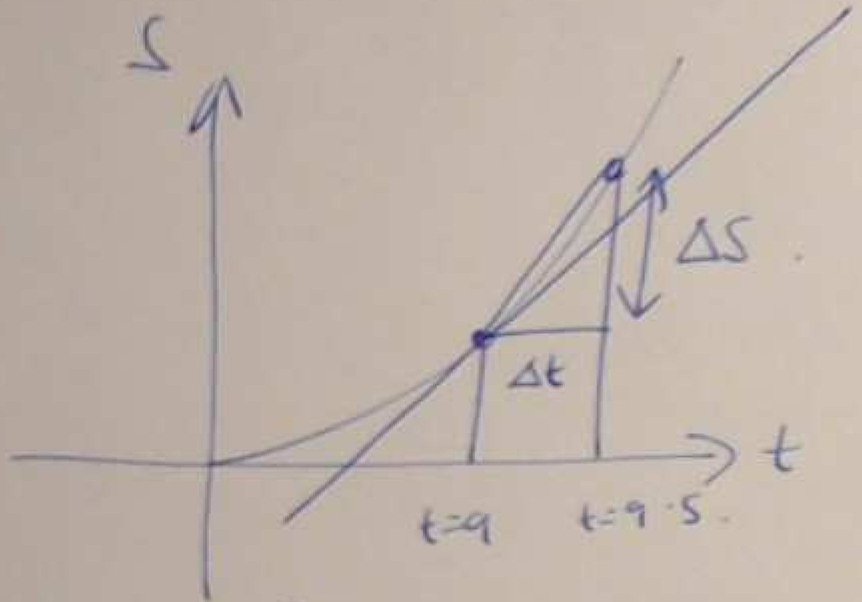




continuous except at $x = 1$.

infinite discontinuity.

$$s(t) = 16t^2.$$



a) $s(9) = 16\times 9^2$.

$s(9.5) = 16\times(9.5)^2$

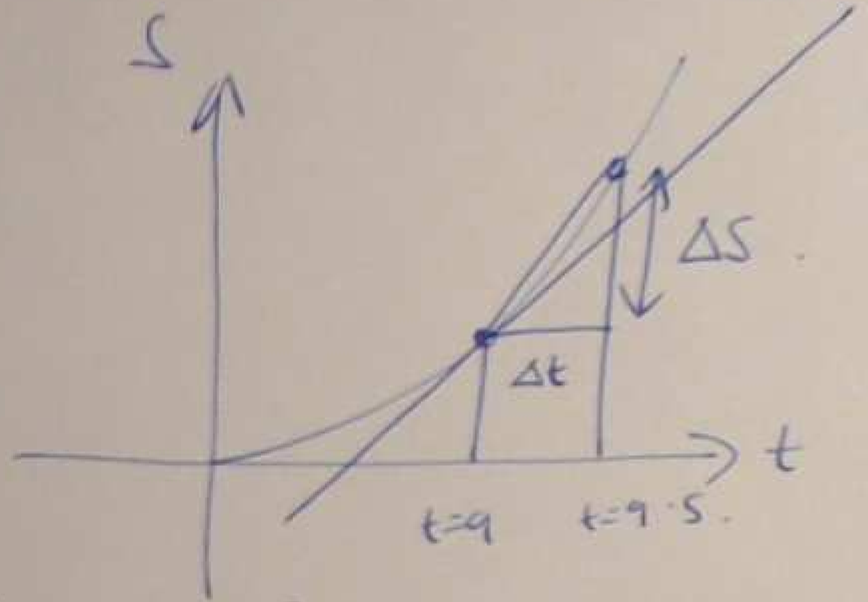
$\Delta s = s(9.5) - s(9) = 16\times(9.5)^2 - 16\times 9^2$.

b)
average speed $= \dfrac{\text{distance}}{\text{time}} = \dfrac{\Delta s}{\Delta t} = \dfrac{16\times(9.5)^2 - 16\times 9^2}{9.5 - 9}$.

c) $\dfrac{s(9+h) - s(9)}{h}$. $\qquad \dfrac{16(9+h)^2 - 16\times 9^2}{h}$.

$h = 0.1$
$h = 0.01$
⋮

$$s(t) = 16t^2.$$



a) $s(9) = 16 \times 9^2.$

$s(9.5) = 16 \times (9.5)^2$

$\Delta s = s(9.5) - s(9) = 16 \times (9.5)^2 - 16 \times 9^2.$

b) average speed $= \dfrac{\text{distance}}{\text{time}} = \dfrac{\Delta s}{\Delta t} = \dfrac{16 \times (9.5)^2 - 16 \times 9^2}{9.5 - 9}.$

c) $\dfrac{s(9+h) - s(9)}{h}$ $\qquad \dfrac{16(9+h)^2 - 16 \times 9^2}{h}$ $\qquad h = 0.1$, $h = 0.01$, $\vdots$

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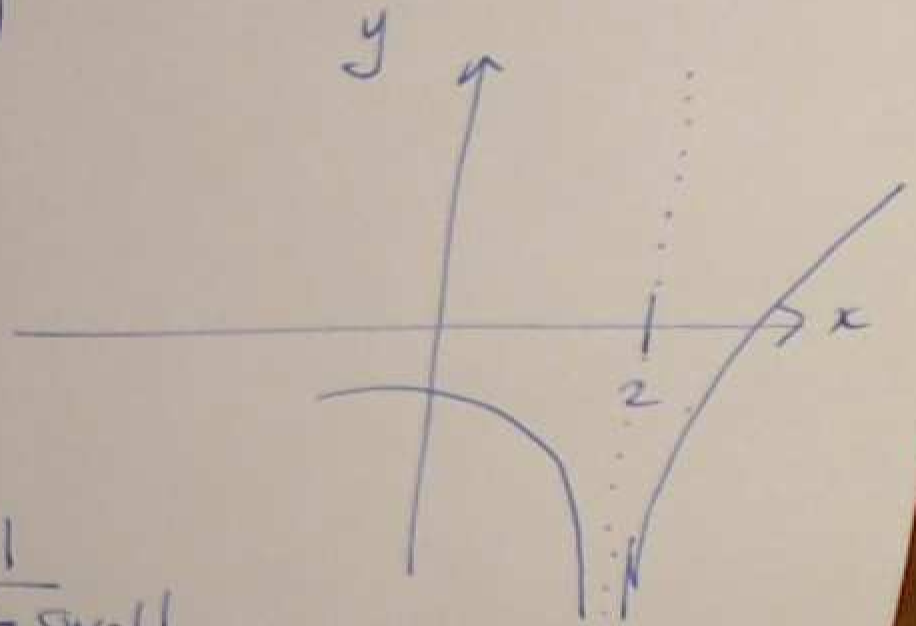
④

WW 2.4 Q2. $f(x) = \frac{x-3}{|x-2|}$



recall if $f(x), g(x)$ cts

then $\frac{f(x)}{g(x)}$ is cts except when $g(x) = 0$.

$x \to 2$

$x > 2$. $\frac{x-3}{x-2}$. $\frac{-ve}{+ve}$. $\frac{-1}{\text{small}}$

$x < 2 = \frac{x-3}{2-x}$ $\frac{-ve}{+ve}$. $\frac{-1}{\text{small}}$.

$x \to 2$

infinite discontinuity

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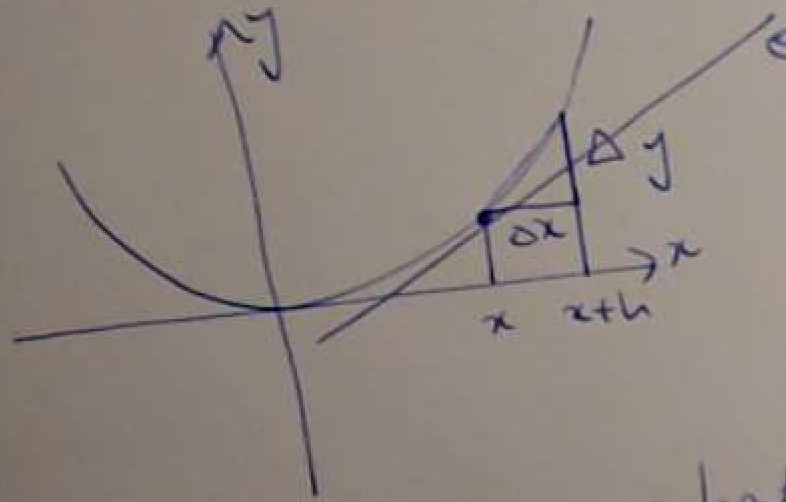
<u>recall</u> want : find rates of change / slopes.



$\Leftarrow$ slope of tangent line.

average rate of change from x to $x+h$. $\frac{\Delta y}{\Delta x}$.

see what happens as h gets small.

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§ 2.5 Evaluating limits algebraically

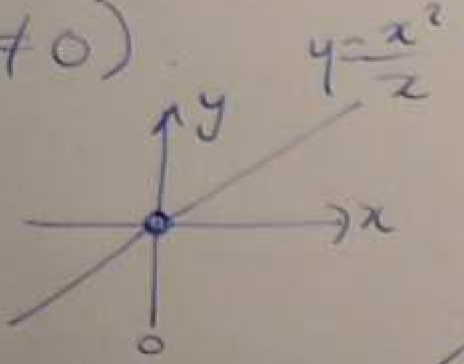
Example $\frac{x^2}{x}$ ← not defined at $x=0$.

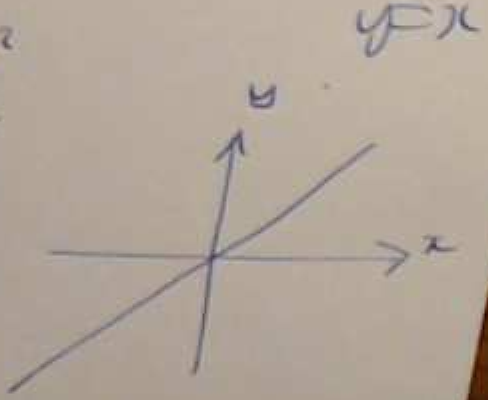
at $x=0$ $\frac{0}{0}$ indeterminate form.

but $\lim_{x\to 0} \frac{x^2}{x}$. does not depend on the value at $x=0$, only close to 0.

$$\frac{x^2}{x} = \frac{x}{1} = x \quad (x \neq 0).$$

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$





indeterminate form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \times 0$, $\infty - \infty$, 0^0.

note $\frac{1}{0}$ not indeterminate – vertical asymptote limits to $\pm\infty$.

Example 1 $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}$.

plug in $x = 3$. $\frac{9 - 12 + 3}{9 - 3 + 12}$.

factor: $\frac{(x-3)(x-1)}{(x-3)(x+4)} = \frac{x-1}{x+4}$. $\frac{6}{0}$.

$(x \neq 3)$

indeterminate form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \times 0$, $\infty - \infty$, 0^0.

note $\frac{1}{0}$ not indeterminate – vertical asymptote
limits to $\pm\infty$.

Example 1

$$\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 + x - 12}.$$

plug in $x = 3$.

$$\frac{9 - 12 + 3}{9 - 3 + 12}.$$

factor: $$\frac{(x-3)(x-1)}{(x-3)(x+4)} = \frac{x-1}{x+4}. \qquad \frac{6}{0}.$$

$(x \neq 3)$

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(8)

$$\lim_{x \to 3} \frac{x^2-4x+3}{x^2+x-12} = \lim_{x \to 3} \frac{x-1}{x+4} = \frac{3-1}{3+4} = \frac{2}{7}.$$

(2) $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$ $\quad x=4: \quad \frac{2-2}{4-4} = \frac{0}{0} \quad ?$

difference of two squares $\quad x-4 = (\sqrt{x}-2)(\sqrt{x}+2).$

$$\frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} \underset{x \neq 4}{=} \frac{1}{\sqrt{x}+2}.$$

$$\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x \to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$$

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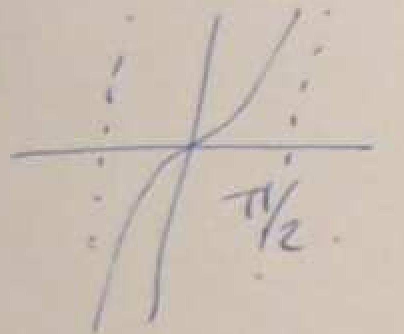
(3) $\lim_{x \to \frac{\pi}{2}} \frac{\tan(x)}{\sec(x)}$ $\quad \frac{\pm\infty}{\pm\infty}$ indeterminate forms.



$$\frac{\tan(x)}{\sec(x)} = \frac{\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)}} = \sin(x)$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan(x)}{\sec(x)} = \lim_{x \to \frac{\pi}{2}} \sin(x) = \sin\left(\frac{\pi}{2}\right) = 1.$$

(4) $$\lim_{x \to 1} \frac{1}{x-1} - \frac{2}{x^2-1} \quad (*) \qquad \infty - \infty \quad ?$$ (10)

$$\frac{1}{x-1} - \frac{2}{x^2-1} = \frac{1}{(x-1)}^{(x+1)} - \frac{2}{(x-1)(x+1)}$$

$\times (x+1)$

$$= \frac{(x+1) \;-2}{(x-1)(x+1)} = \frac{x-1}{(x-1)(x+1)}$$

$$(x \neq 1) \quad = \frac{1}{x+1} \qquad (*) = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$$

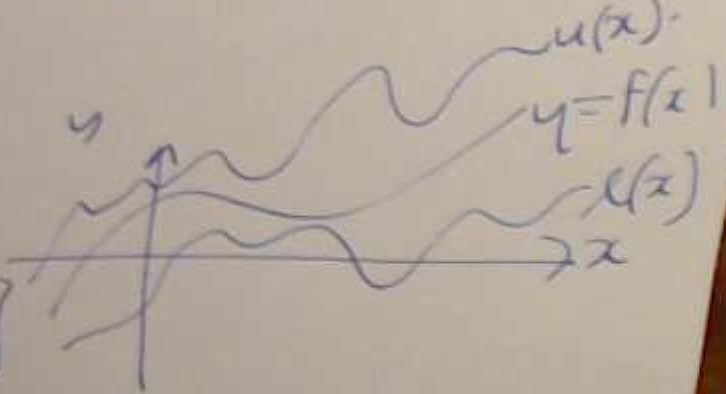
§2.6 Trigonometric limits

Q: what is $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x}$?

plug $x=0$: $\dfrac{0}{0}$?

A: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$. (approx, try $x=0.1$, $x=0.01$, ...)

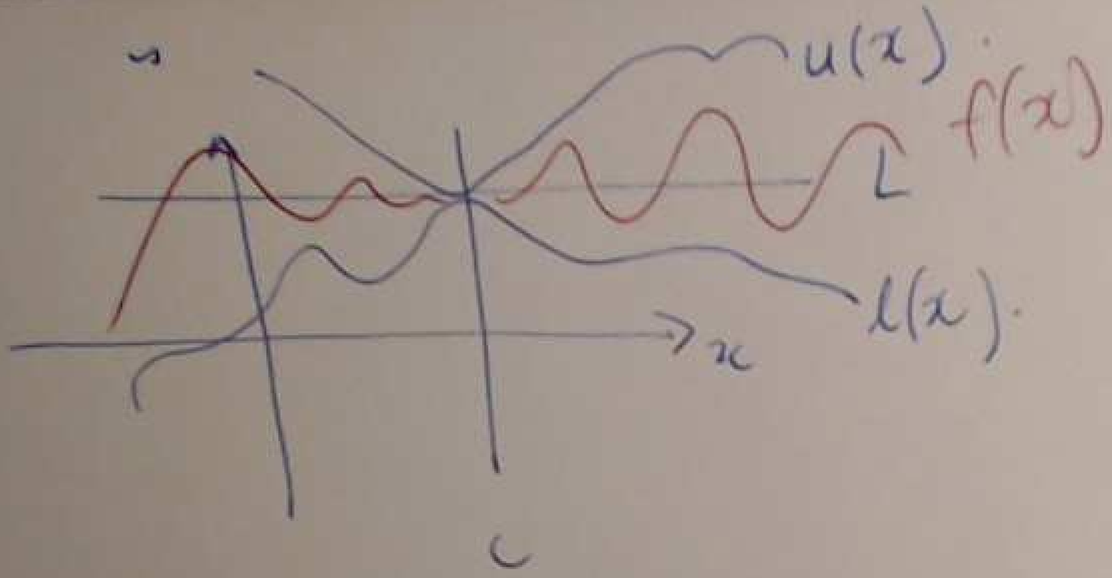
squeeze theorem

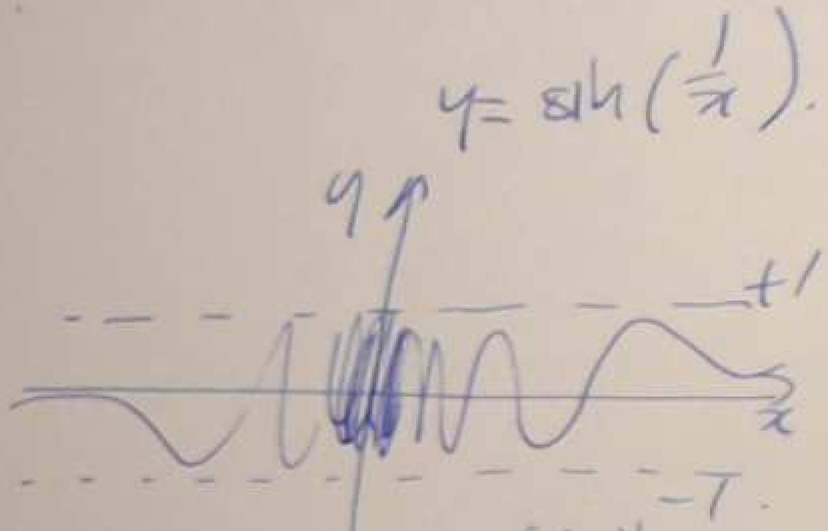


[suppose $l(x) \le f(x) \le u(x)$]

Thm: if $\lim\limits_{x\to c} l(x) = L = \lim\limits_{x\to c} u(x)$

then $\lim\limits_{x\to c} f(x) = L$



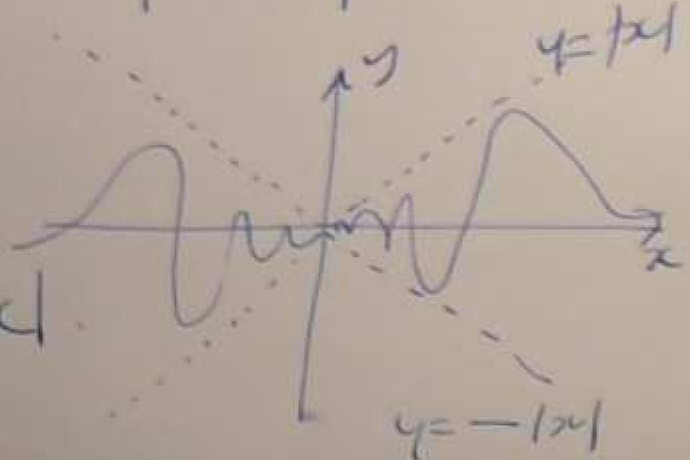


no limit at $x = 0$

Example $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$

$|\sin(x)| \leq 1$

$\left| x \sin\left(\frac{1}{x}\right) \right| \leq |x|$

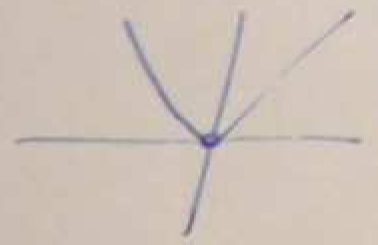


$-|x| \leq x \sin\left(\frac{1}{x}\right) \leq |x|$

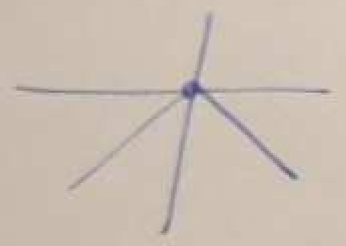
$|x|$ upper bound

$-|x|$ lower bound

$$\lim_{x \to 0} |x| = 0$$



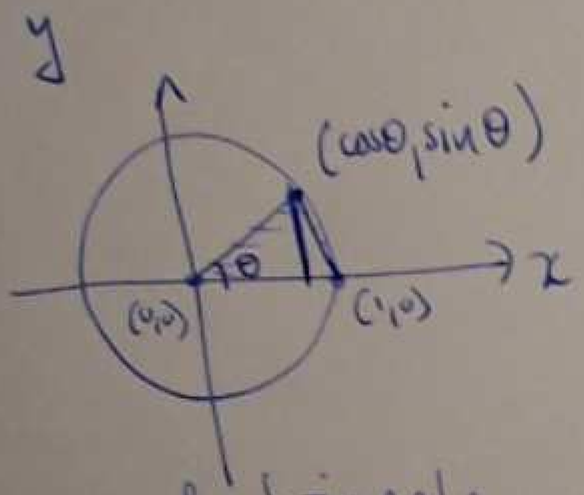
$$\lim_{x \to 0} -|x| = 0$$

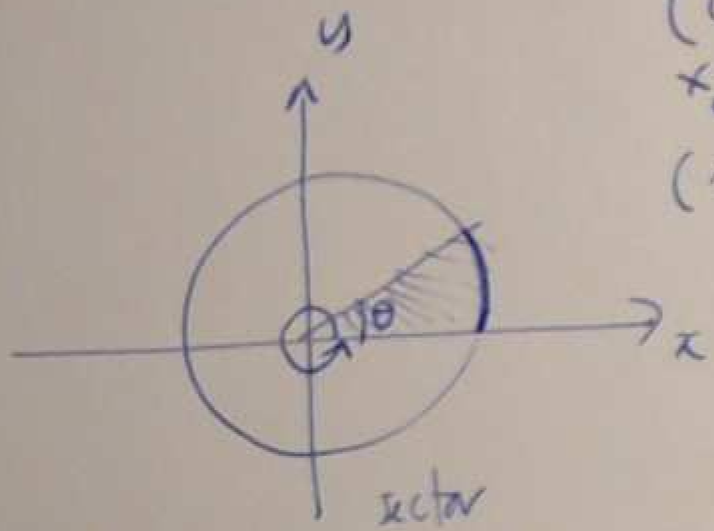


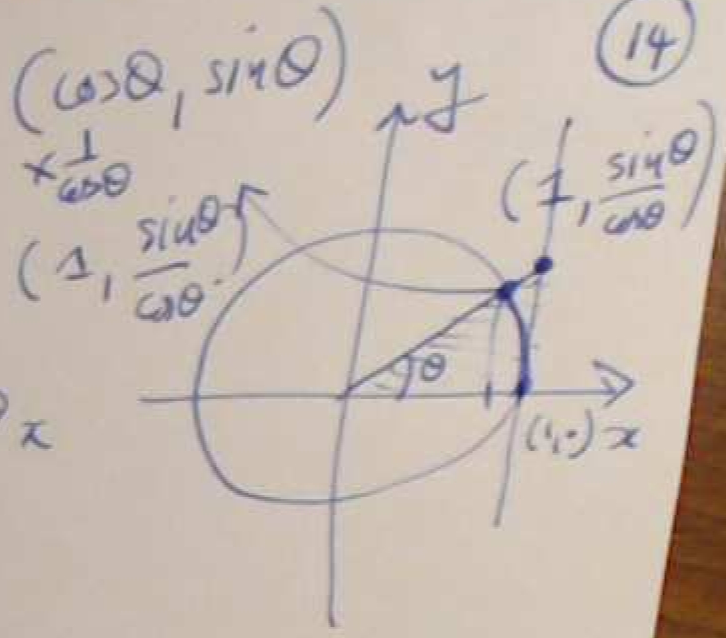
$$\Rightarrow \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Thm $\displaystyle \lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1-\cos x}{x} = 0.$

Proof: $\displaystyle \frac{\sin \theta}{\theta} \qquad 0 < \theta < \frac{\pi}{2}$







area of triangle

$\frac{1}{2}$ base $\times$ height.

$\frac{1}{2} \times 1 \times \sin\theta$

area. $r=1$

$\pi r^2 = \pi$.

$\pi \frac{\theta}{2\pi}$

$\frac{1}{2} bh$.

$\frac{1}{2} \times 1 \times$.

$$\frac{\sin\theta}{2} \leq \frac{\theta}{2} \leq \frac{1}{2} \frac{\sin\theta}{\cos\theta}.$$

$$\sin\theta \leq \theta \leq \frac{\sin\theta}{\cos\theta}.$$

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(15)

$$\underbrace{\sin\theta \le \theta}_{\displaystyle \frac{\sin\theta}{\theta} \le 1} \qquad \theta \le \frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\theta} \le 1 . \qquad \cos\theta \le \frac{\sin\theta}{\theta}.$$

$$\underbrace{\cos\theta}_{l(\theta)} \le \frac{\sin\theta}{\theta} \le \underbrace{1}_{u(\theta)}.$$

apply squeeze theorem

$$\lim_{\theta\to 0} 1 = 1 \qquad \lim_{\theta\to 0} \cos\theta = 1.$$

$$\Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1. \quad \square$$

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Thm $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$

Examples ① $\lim_{x \to 0} \frac{\sin(3x)}{2x}$.

set $3x = \theta$

$x = \theta/3$.

know $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

$\lim_{x \to 0} \frac{\sin(\theta)}{2(\theta/3)} = \lim_{x \to 0} \frac{3}{2} \frac{\sin\theta}{\theta}$

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Thm $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ $\qquad$ $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$

Examples 1 $\lim_{x \to 0} \frac{\sin(3x)}{2x}$

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$$\lim_{x \to 0} \frac{\sin(\theta)}{2(\theta/3)} = \lim_{x \to 0} \frac{3}{2} \frac{\sin\theta}{\theta}$$

$$\lim_{\theta\to 0}\frac{3}{2}\frac{\sin\theta}{\theta} = \frac{3}{2}\underbrace{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}_{1} = \frac{3}{2}.$$

Ex 2

$$\lim_{t\to 0}\frac{1-\cos t}{\sin t} = \lim_{t\to 0}\frac{1-\cos t}{t}\cdot\frac{t}{\sin t}$$

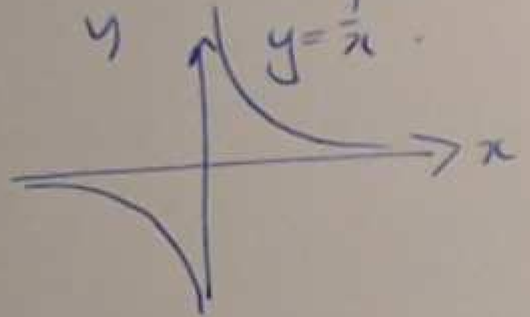
$\frac{0}{0}$

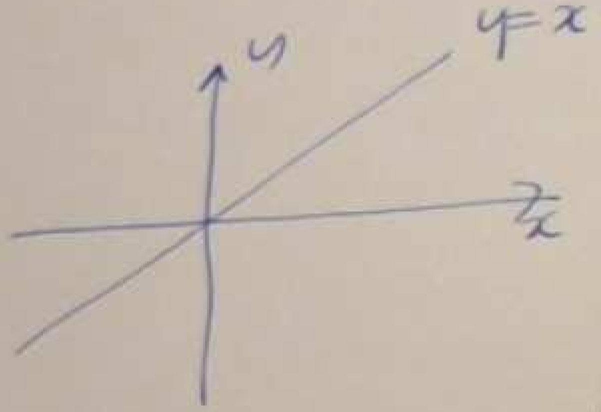
$$= \underbrace{\lim_{t\to 0}\frac{1-\cos t}{t}}_{0\ (\text{Thm})}\cdot\underbrace{\lim_{t\to 0}\frac{t}{\sin t}}_{=\frac{1}{\underbrace{\lim_{t\to 0}\frac{\sin t}{t}}_{1\ (\text{Thm})}}}\ 1$$

$$= 0.$$

§ 2.7 Limits at infinity

key observation

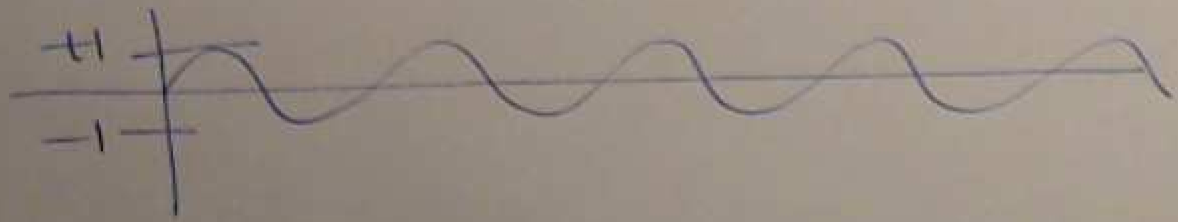




$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

$$\lim_{x\to\infty} x = +\infty.$$

bad example $\lim_{x\to\infty} \sin(x) \leftarrow$ DNE.



① $$\lim_{x\to\infty} \frac{3x}{2x-1} = \lim_{x\to\infty} \frac{3}{2-1/x}$$

$$= \cancel{\lim} = \frac{3}{2 - \underbrace{\lim_{x\to\infty} 1/x}_{0}} = \frac{3}{2}$$

② $$\lim_{x\to\infty} \frac{x^2+x}{x-3} = \lim_{x\to\infty} \frac{x+\underbrace{1/x}_{\to 0}}{1-\underbrace{3/x}_{\to 0}}$$

$$= \lim_{x\to\infty} x = \infty$$