HW1 Q8

x^2+4x-3

complete the square

$$x^2+4x-3$$

$$(x+2)^2-7$$

$$x^2+4x+4-7$$

minimum value is -7

where does it occur? $x=-2$

$-x^2+\cdots$

Unpin Video Recording...

WW1Q7 2

$x - x^2 + 2$

$-x^2 + x + 2$

$-(x^2 - x - 2)$

$-(x-2)(x+1)$

$x = 2, -1$

$4x^2 - 5x - 4$

1, 4   1, 4

2, 2   2, 2

$(2x - 1)(2x + 4)$

$4x^2 + 6x - 4$ ✗

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{5 \pm \sqrt{25 - 4 \cdot 4 \cdot -4}}{8}$$

$$\frac{5 \pm \sqrt{25 + 64}}{8}$$

$$\frac{5 \pm \sqrt{91}}{8}$$

$$\frac{5 + \sqrt{91}}{8}, \quad \frac{5 - \sqrt{91}}{8}$$

$-1, 1]$

$[0, \pi]$

$f(x) = b^x$ $(b > 0)$

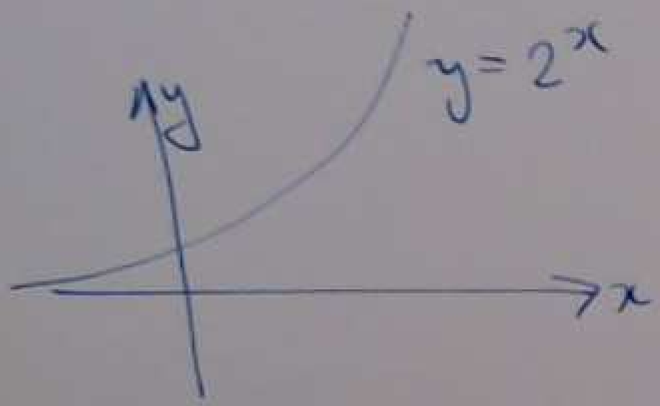
polynomial x^n

Joseph Maher

§1.6 Exponential and logarithm functions

Example $x \mapsto 2^x$.

x	-2	1	0	1	2	3	4
$f(x)$	$1/4$	$1/2$	1	2	4	8	16



- can use any positive number instead of 2

$f(x) = b^x$ b called the base.

useful properties

- positive $b^x > 0$
- b^x increasing if $b > 1$
  decreasing if $b < 1$

$b > 1$

- b^x grows faster than any polynomial x^n.

<u>exponent rules</u>:

$$b^0 = 1$$

$$b^x b^y = b^{x+y}$$

$$b^{-x} = \frac{1}{b^x}$$

$$\frac{b^x}{b^y} = b^{x-y}$$

$$(b^x)^y = b^{xy}$$

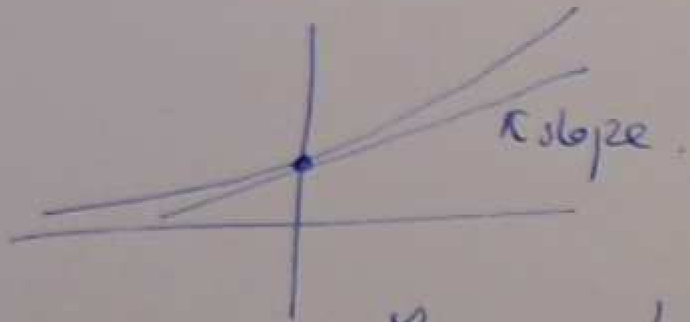
$$b^{1/n} = \sqrt[n]{b}$$

- there is a special base $b = e = 2.71828\ldots$

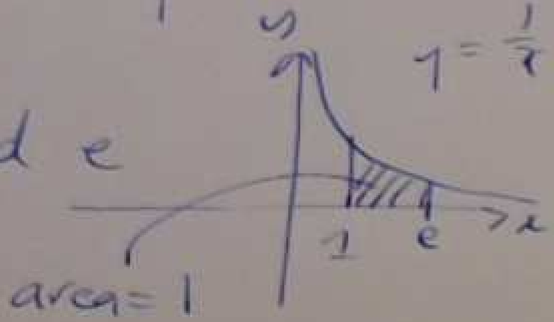
$f(x) = e^x$ natural exponential function.

<u>key properties</u>

(1) e is the unique number such that e^x has slope 1 at $x = 0$

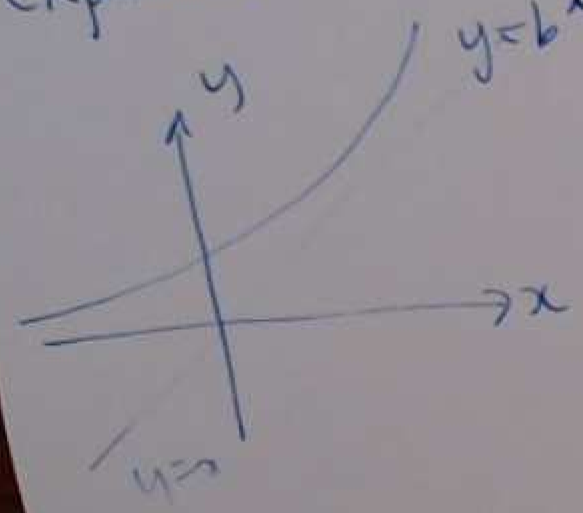


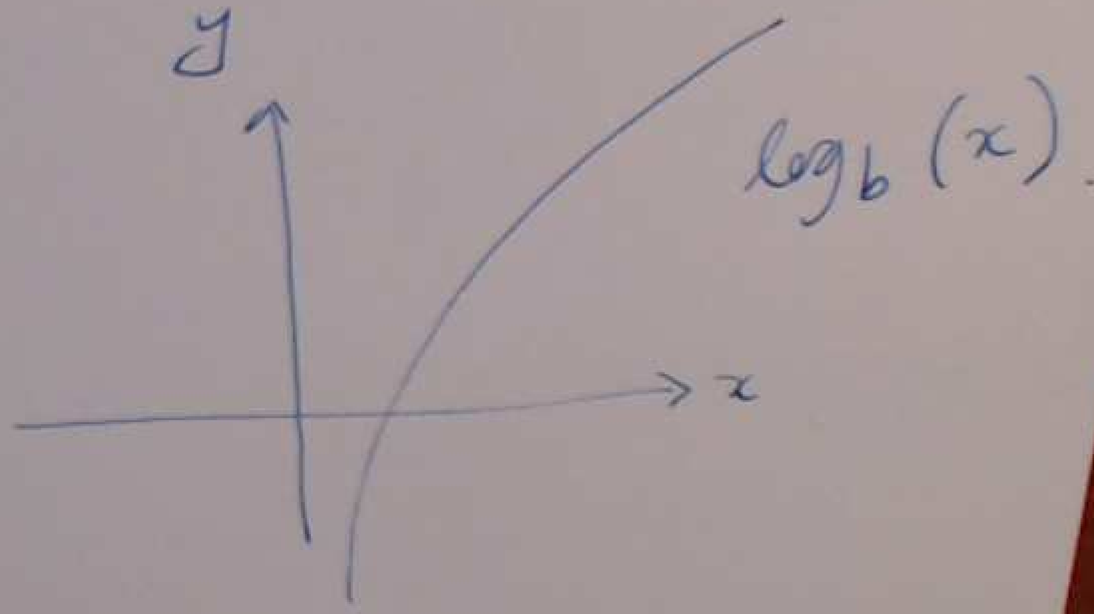
(2) e is the unique number such that the area under the graph of $y = \frac{1}{x}$ between 1 and e has area $= 1$.



logarithms

the logarithm is the inverse function of the exponential function.





the natural logarithm $\ln(x) = \log_e(x)$.

inverse function properties

$f^{-1}(f(x)) = x = f(f^{-1}(x))$

so $b^{\log_b(x)} = x = \log_b(b^x)$

- log rules: $\log_b(1) = 0 \qquad \log_b(b) = 1$

$$\log_b(st) = \log_b(s) + \log_b(t)$$

$$\log_b\left(\frac{1}{t}\right) = -\log_b(t)$$

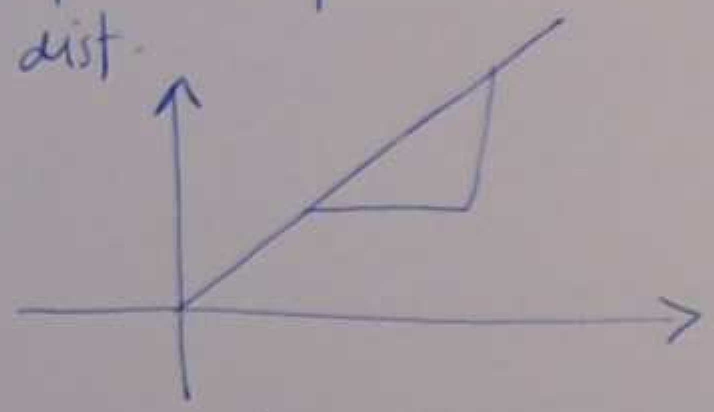
$$\log_b(s^t) = t\log_b(s)$$

$$\boxed{\log_b(x) = \frac{\log_a(x)}{\log_a(b)}}$$

base conversion rule

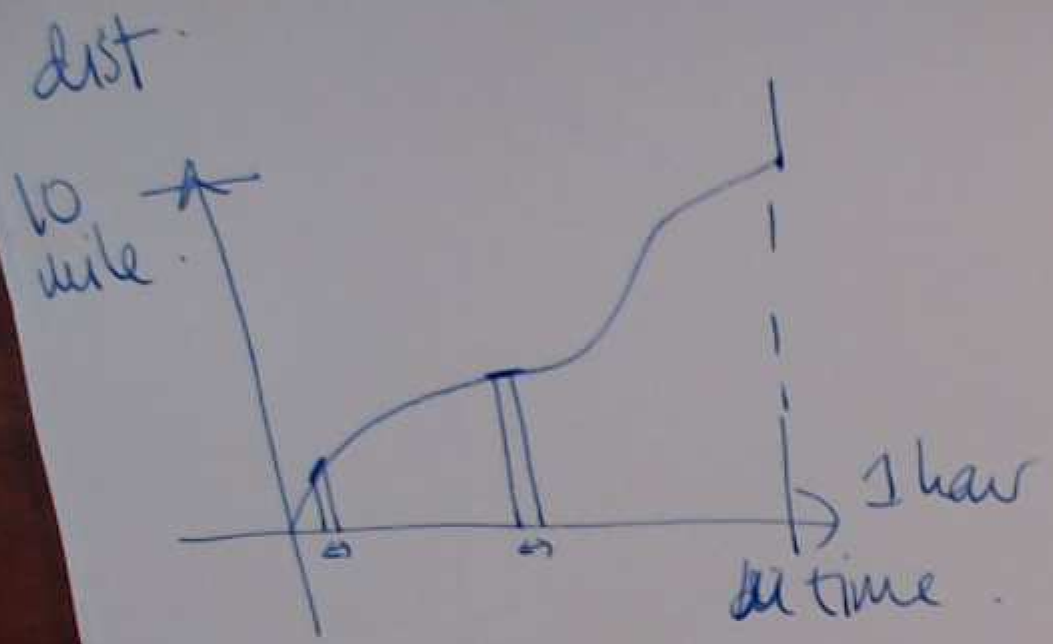
§ 2.1 Limits, rates of change, tangent lines

<u>motivation</u> : velocity, example : driving at constant speed.



$$\text{velocity} = \frac{\text{distance}}{\text{time}} = \text{slope of the line.}$$

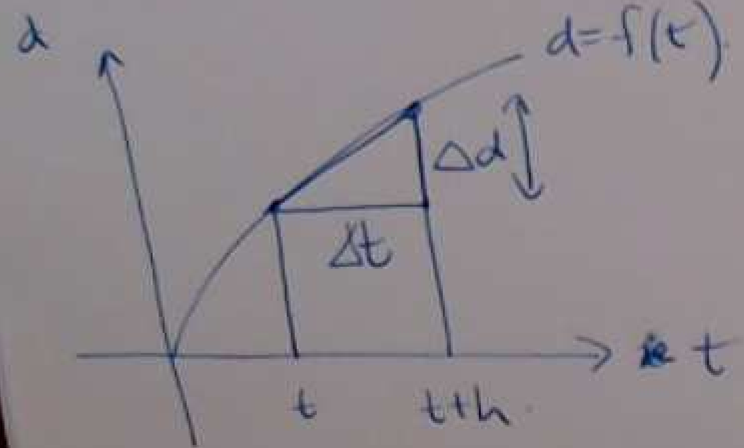
<u>problem</u> : what happens if you don't travel at constant speed?



$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{10}{1} = 10\text{mph}$$

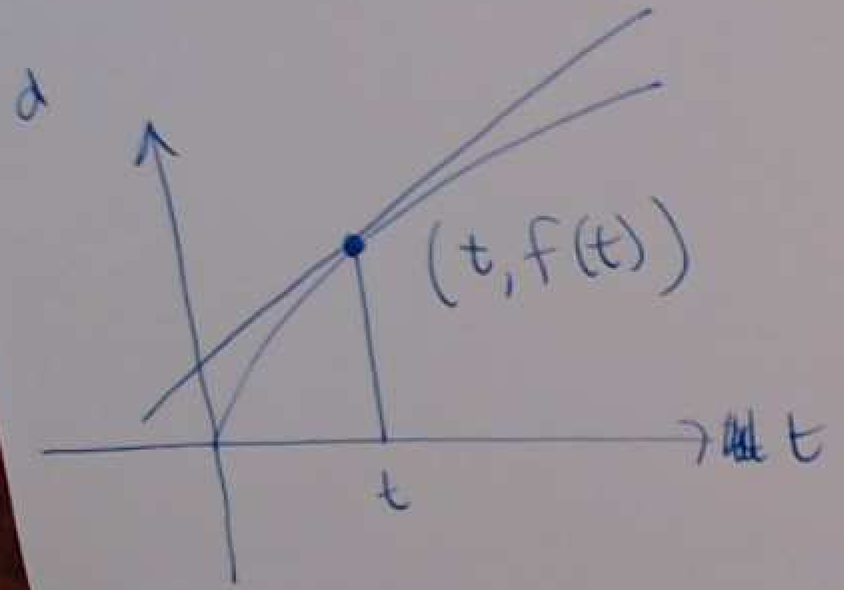
we can look at the average speed over any time interval, even very short ones.



average speed on interval $[t, t+h]$

$$\frac{\Delta d}{\Delta t} = \frac{f(t+h) - f(t)}{(t+h) - t} = \frac{f(t+h) - f(t)}{h}$$

Q: what is the speed at time t?
(sometimes called the instantaneous speed / rate of change)



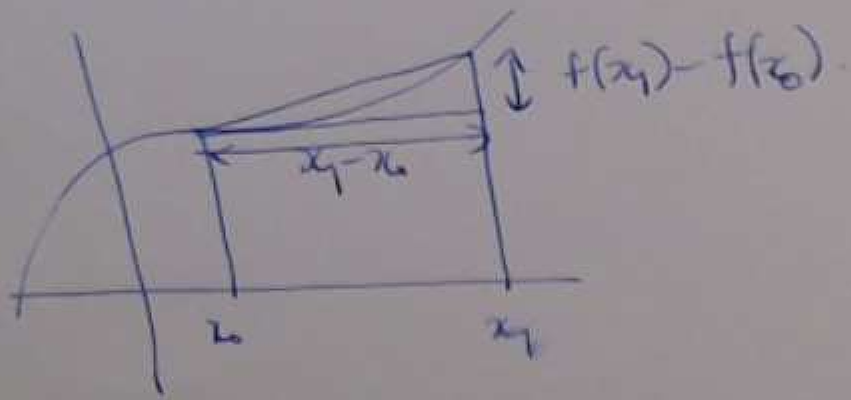
A: speed is the slope of the tangent line at $(t, f(t))$.

<u>idea / hope</u>: as the length of the interval $[t, t+h]$ gets small, the average speed gets close to the speed at t. <u>This works</u> for "nice" functions.

<u>observation</u>: this works for any function $y = f(x)$.
not just speed.

<u>summary</u> average rate of change over an interval $[t, t+h]$.

$[x_0, x_1] \leftarrow$ $\quad \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$.



§ 2.2. Limits

aim: find slope of tangent line.

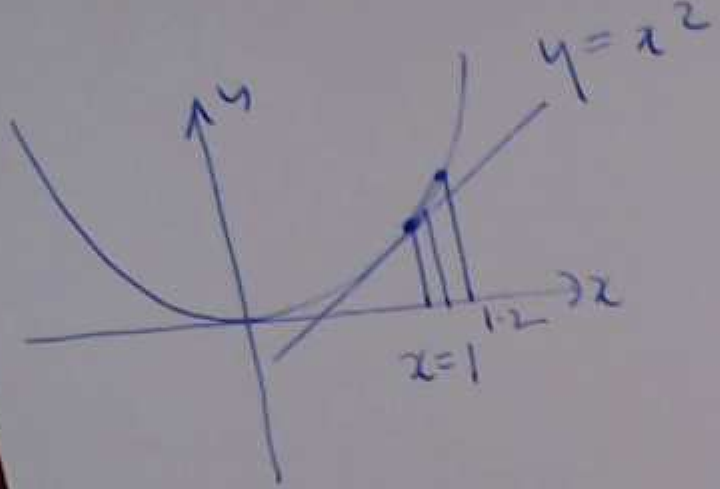
Know: average rate of change: $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$.

Q: why can't we just set $x_0 = x_1$?

A: doesn't work $\dfrac{f(x_0) - f(x_0)}{x_0 - x_0} = \dfrac{0}{0}$ undefined !!

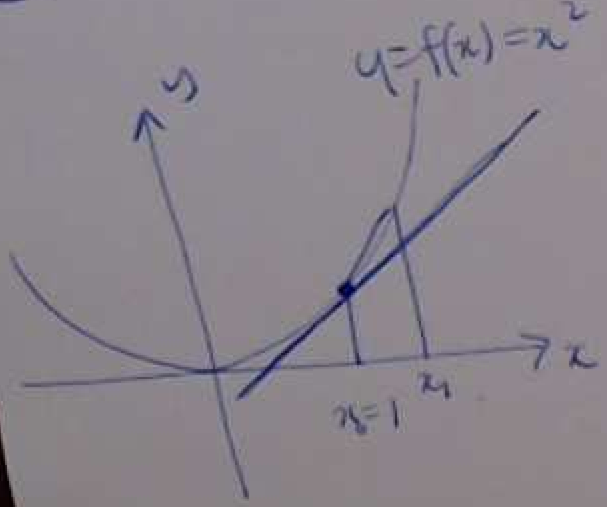
observations

1. if we draw careful pictures, the average slope gets closer to the slope of the tangent line.



hard to draw accurate pictures.

2 seems to work for sample calculations.



$x_1 = 2$: $\dfrac{f(2)-f(1)}{2-1} = \dfrac{4-1}{1} = 3$

$x_1 = 1.1$ $\dfrac{f(1.1)-f(1)}{1.1-1} = \dfrac{1.21-1}{0.1}$

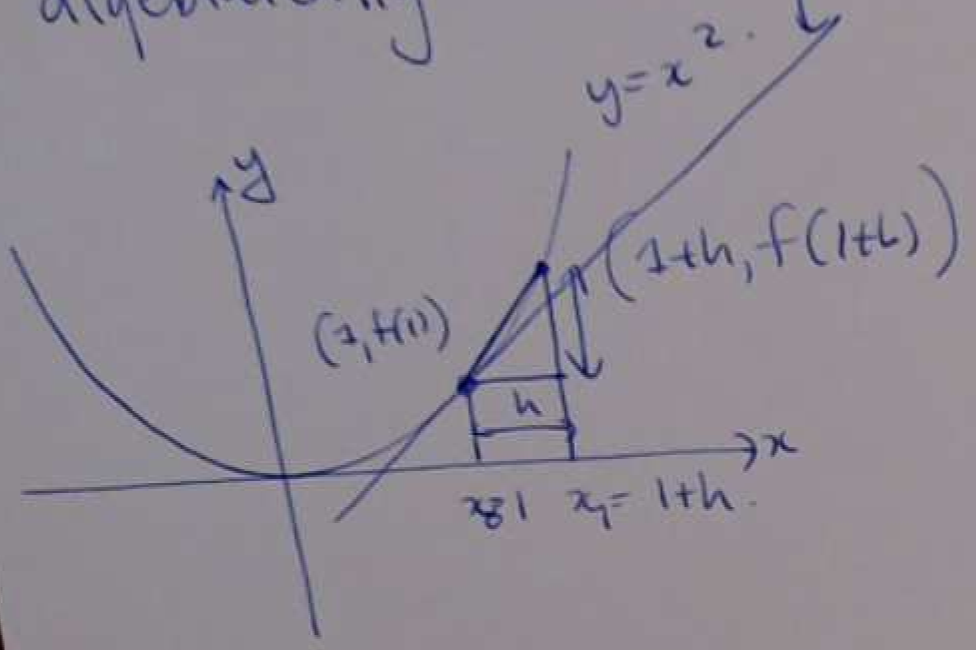
$= 2.1$

$x_1 = 1.01$ $\dfrac{f(1.01)-f(1)}{1.01-1} = \dfrac{1.0201-1}{0.01}$

$= 2.01$

(14)

(3) algebraically: slope 2.



average rate of change from x_0 to x_1

$1 \qquad 1+h$

$$\frac{f(1+h) - f(1)}{1+h-1} = \frac{f(1+h)-f(1)}{h}$$

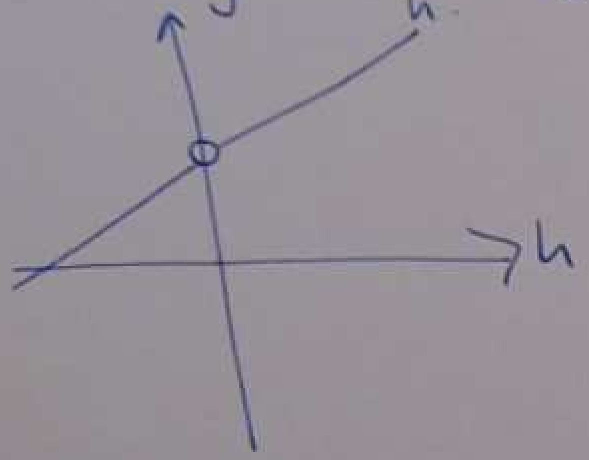
$$= \frac{(1+h)^2 - 1^2}{h} = \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} = \frac{2h+h^2}{h}$$

$= 2+h$ as $h \to 0$

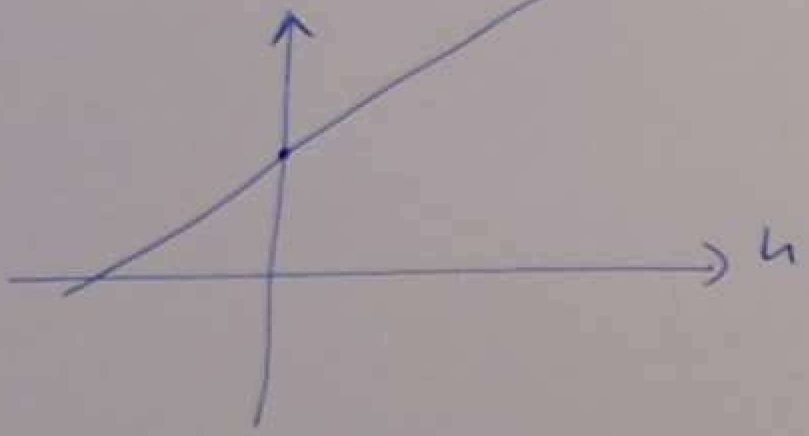
average slope $\longrightarrow 2$

$$\frac{2h+h^2}{h} = 2+h. \qquad (h \neq 0).$$

$y = \dfrac{2h+h^2}{h} \; (h\neq 0).$



$y = 2+h.$



Defn Let f be a function defined on an interval containing c, but not necessarily at c.
We say "the limit of $f(x)$ as x approaches c is equal to L" if $|f(x) - L|$ becomes arbitrarily small as x gets close to c.

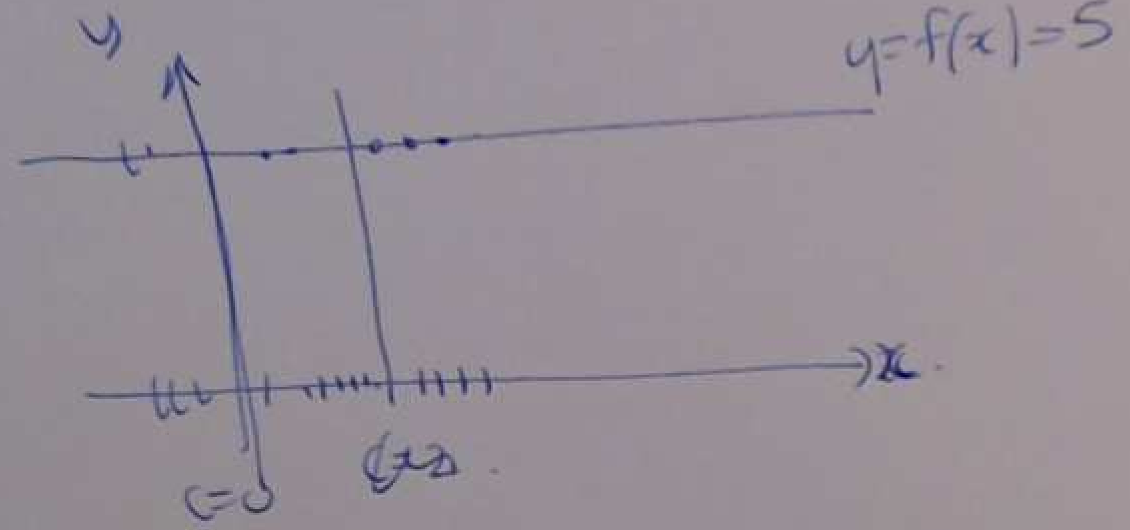
notation $$\lim_{x \to c} f(x) = L$$

or $$f(x) \to L \text{ as } x \to c.$$

we also say "$f(x)$ converges to L as x tends to c".

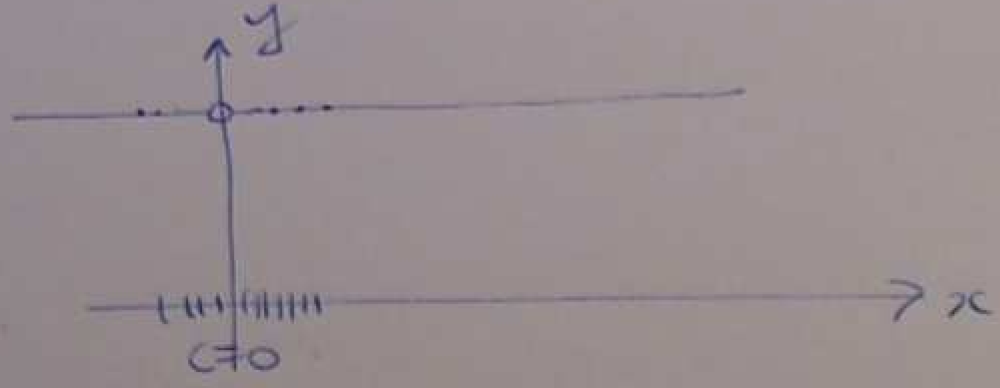
Examples

a) $f(x) = 5, \quad c = 0$.



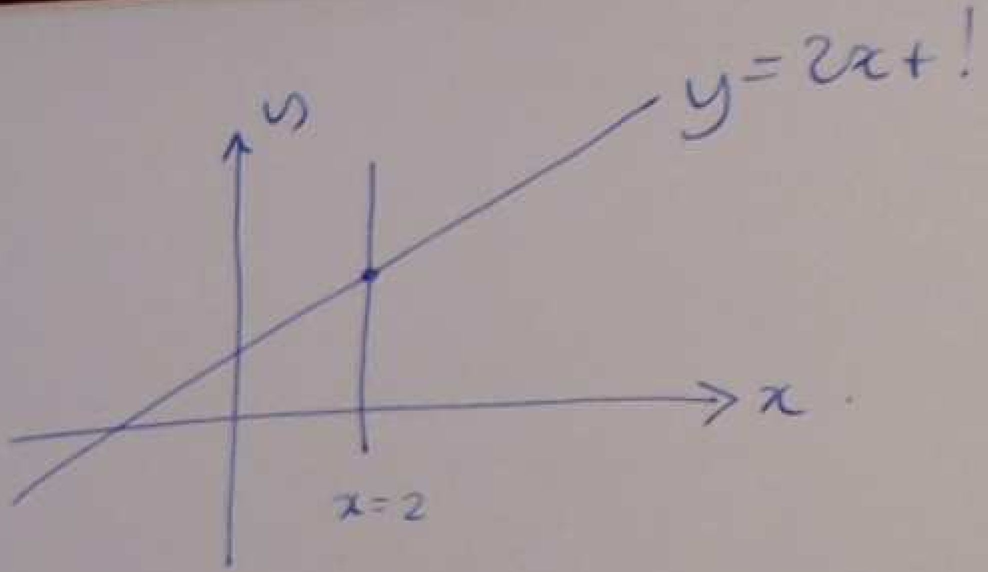
$$\lim_{x\to 0} f(x) = 5.$$

b) $f(x) = \dfrac{5x}{x}$.



$$\lim_{x\to 0} f(x) = 5$$

c) $\lim\limits_{\substack{x \to 2 \\ c}} \underbrace{2x+1}_{f(x)} = \underbrace{5}_{L}$.



way to show that

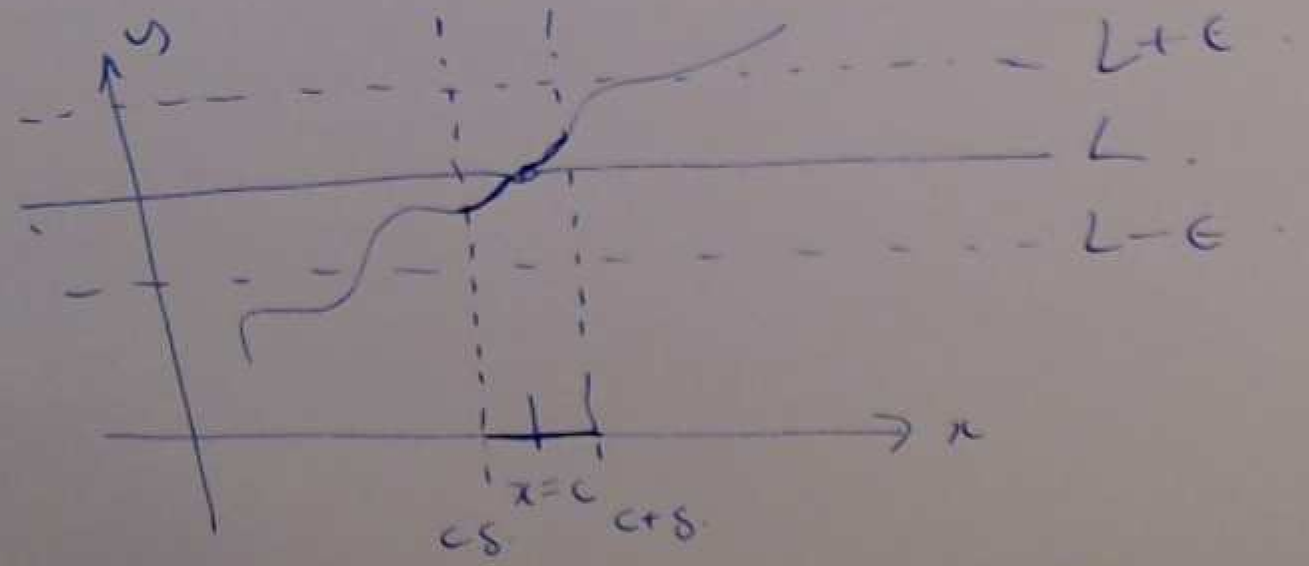
$|f(x) - L|$ is close to 0
when x is close to 2.

$|2x+1-5|$ is close to zero when x is close to 2.

$|2x-4| = |2(x-2)| = 2|x-2|$.

if $|x-2|$ is close to zero, so is $2|x-2|$. ✓

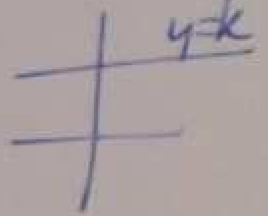
Precise defn Let $f(x)$ be defined an ~~a~~ interval containing c, but nec. at c, we say $\lim_{x \to c} f(x) = L$ if for all $\epsilon > 0$ there is a $\delta > 0$ s.t. if $|x - c| < \delta$ then $|f(x) - L| < \epsilon$.



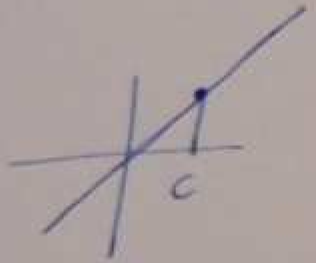
useful facts:

Thm for any constants k, c

$$\lim_{x \to c} k = k.$$



$$\lim_{x \to c} x = c.$$



investigating limits:

try:
- draw a picture
- calculate close values.
- algebra.

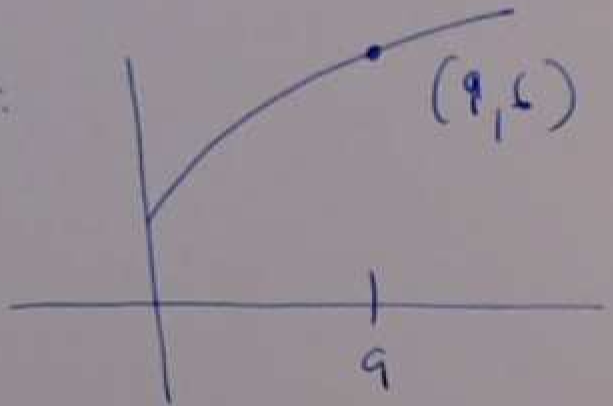
Example $\lim_{x \to 9} \dfrac{x-9}{\sqrt{x}-3}$

problem: plug in $x=9$?

$\dfrac{9-9}{\sqrt{9}-3} = \dfrac{9-9}{3-3} = \dfrac{0}{0}$.

undefined.

- draw picture:



look like $f(9) \approx 6$.

- **calculations**

x	$\dfrac{x-9}{\sqrt{x}-3}$
9.1	6.016
9.01	6.002

· algebraically:

$$\frac{x-9}{\sqrt{x}-3} \quad x \to 9.$$

difference of two squares

$$(a-b)(a+b) = a^2 - \cancel{ab} + \cancel{ab} - b^2$$
$$= a^2 - b^2$$

$$x - 9 = (\sqrt{x})^2 - 3^2.$$
$$= (\sqrt{x}-3)(\sqrt{x}+3).$$

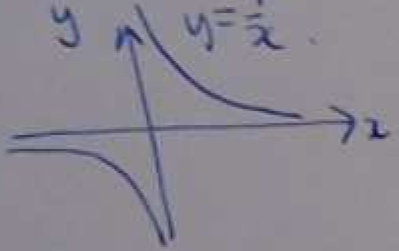
$$\frac{x-9}{\sqrt{x}-3} = \frac{\cancel{(\sqrt{x}-3)}(\sqrt{x}+3)}{\cancel{(\sqrt{x}-3)}} = \sqrt{x}+3.$$

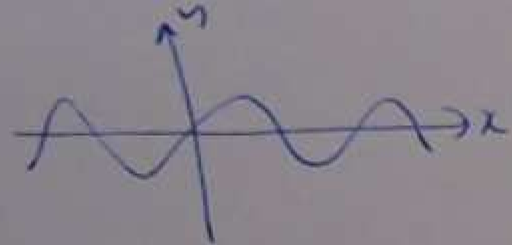
$x \neq 9.$

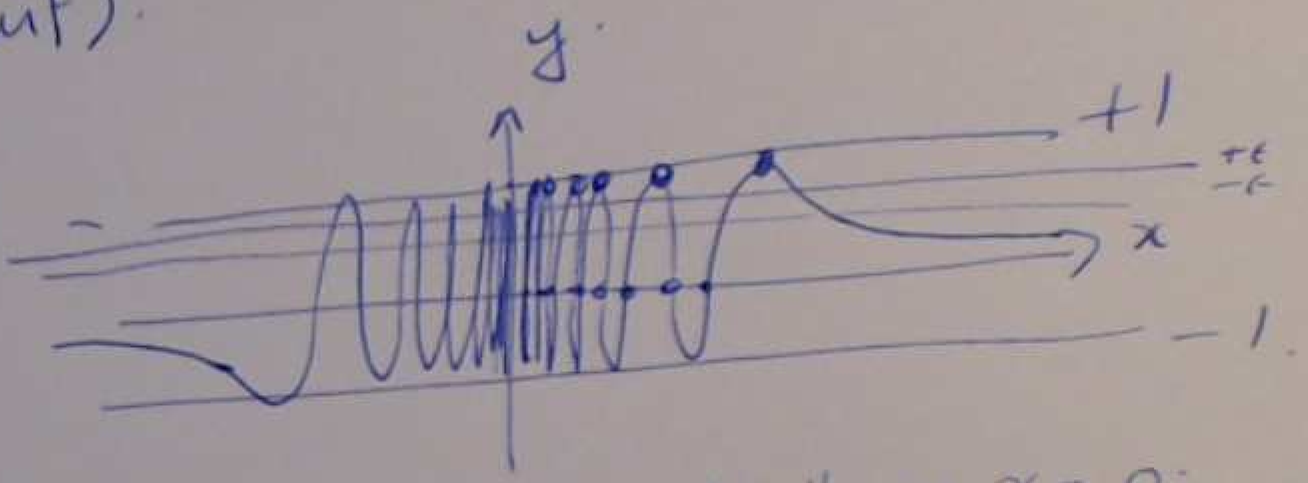
$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \to 9} \sqrt{x}+3 = 6$$

Bad example (no limit).

$$f(x) = \sin\left(\frac{1}{x}\right)$$







↑ no limit as $x = 0$.

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = \text{DNE}.$$

does not exist

note:

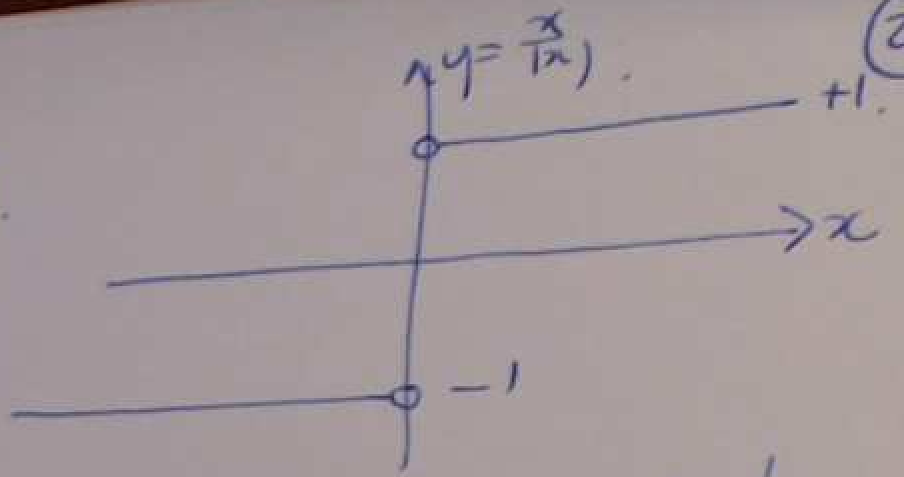
$n \in \mathbb{N}$. $\{1, 2, 3, \dots\}$

$$f\left(\frac{1}{2\pi n}\right) = \sin\left(\frac{1}{\cancel{1/}2\pi n}\right) = \sin(2\pi n) = 0$$

$$f\left(\frac{1}{2\pi n + \pi/2}\right) = \sin\left(2\pi n + \frac{\pi}{2}\right) = 1$$

one sided limits

example $f(x) = \frac{x}{|x|}$.



$$f(x) = \frac{x}{|x|} = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

sometimes useful to distinguish left limit / right limit / two sided limit.

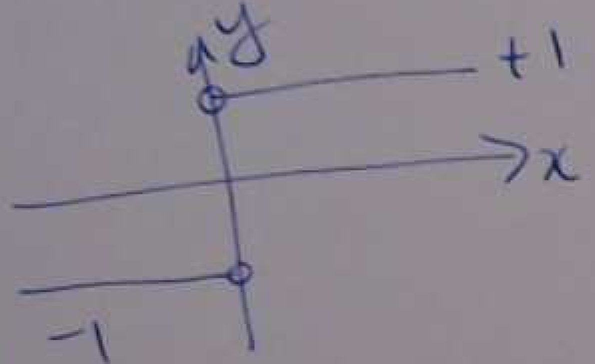
notation $\lim_{x \to 0+} f(x)$ means right limit ($x > 0$).

$\lim_{x \to 0-} f(x)$ means left limit ($x < 0$)

<u>note</u> if the two-sided limit exists then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x)$$

example $f(x) = \frac{x}{|x|}$.

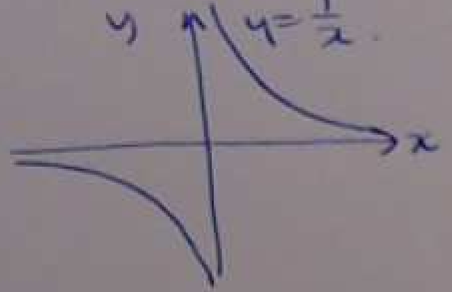


$$\left.\begin{aligned}\lim_{x\to 0+} f(x) &= +1 \\ \lim_{x\to 0-} f(x) &= -1\end{aligned}\right\} \; 1 \neq -1$$

$\Rightarrow \lim_{x\to 0} \frac{x}{|x|}$ DNE

Infinite limits.

$f(x) = \frac{1}{x}$



$\lim_{x\to 0+} f(x) = \frac{1}{x} = +\infty$.

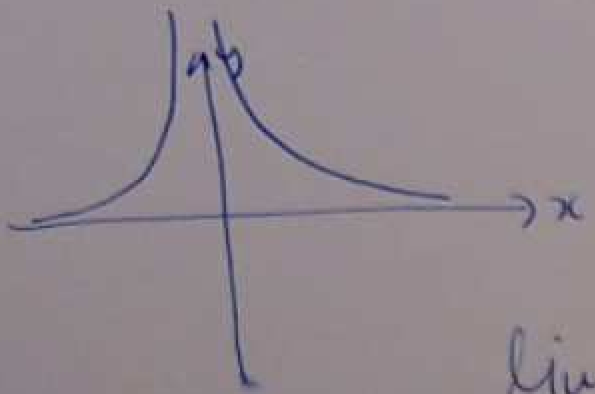
$\lim_{x\to 0-} \frac{1}{x} = -\infty$.

we say $\lim_{x \to c} f(x) = +\infty$ if $f(x)$ becomes arbitrarily large and positive as $x \to c$.

$\lim_{x \to c} f(x) = -\infty$ if $f(x)$ becomes negative and large in magnitude as $x \to c$.

Example

$f(x) = \frac{1}{x^2}$



$$\lim_{x \to 0} \frac{1}{x^2} = +\infty.$$