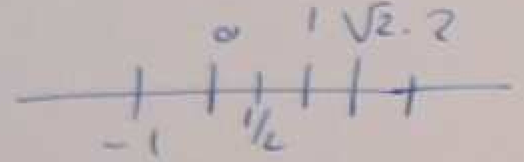
MTH 231

Joseph Maher
joseph.maher@csi.cuny.edu

office hours M 5:30 – 6:20   W 4:40 – 6:20.

§1.2 Linear and quadratic functions

recall: input set (domain) $\xrightarrow{\text{function}}$ output set (range).



example: $f : \mathbb{R} \longrightarrow \mathbb{R}$. $\mathbb{R} \leftarrow$ real numbers.

$x \longmapsto x^2$ $f(x) = x^2$.

$3 \longmapsto 9$.
$-2 \longmapsto 4$
$1/2 \longmapsto 1/4$

notation: $f : \mathbb{R} \longrightarrow \mathbb{R}$. rule: $f(x) = x^2$.

name ↗ f, ↑ domain, ↖ range.

<u>Example</u> $+ : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$$(a,b) \longmapsto (a+b)$$

evaluation at 0: $\{\text{functions } f : \mathbb{R} \to \mathbb{R}\} \longrightarrow \mathbb{R}$

$$f \longmapsto f(0).$$

<u>key property</u> every input goes to exactly one output, <u>not</u> a collection of outputs.

$$f : \mathbb{R} \longrightarrow \mathbb{R}.$$
$$1 \longmapsto 4.$$

$$\cancel{f : \mathbb{R} \longrightarrow \{-1, 1\}.}$$
$$\cancel{1 \longmapsto}$$

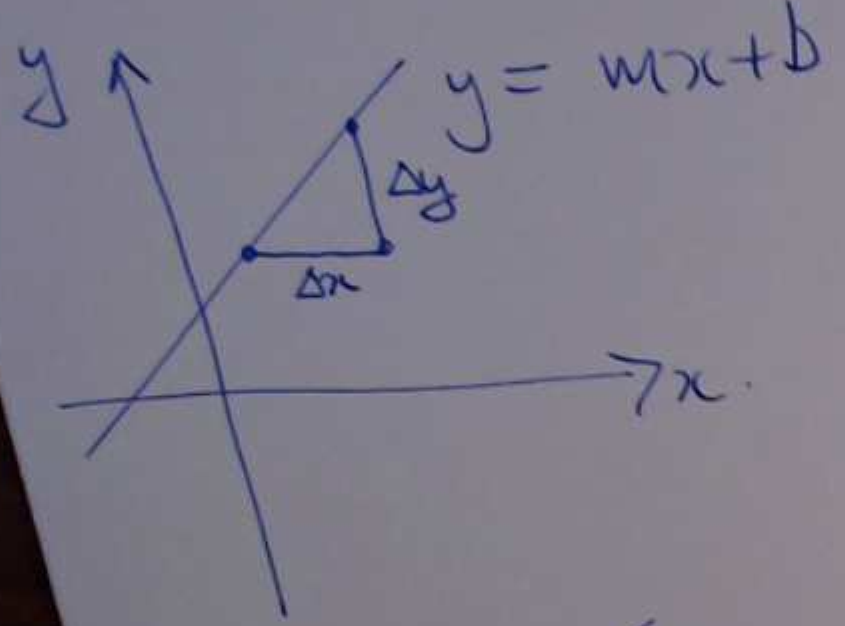
$f(x) = x + 3.$

A <u>linear function</u> $f : \mathbb{R} \to \mathbb{R}$ has the form

$$f(x) = mx + b$$

m, b are "constants", i.e. don't depend on x.

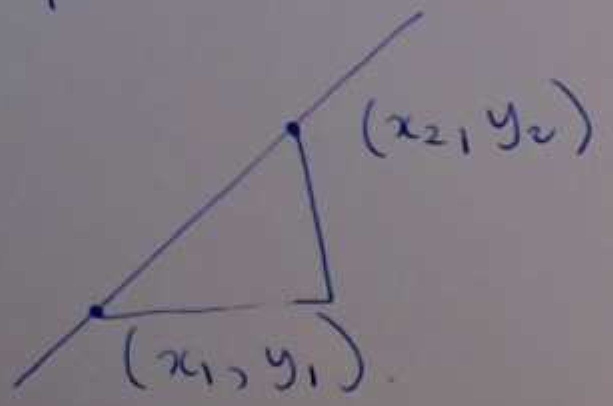
the graph of a linear function is a straight line.



$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}}$$

$$= \frac{\text{rise}}{\text{run}}$$

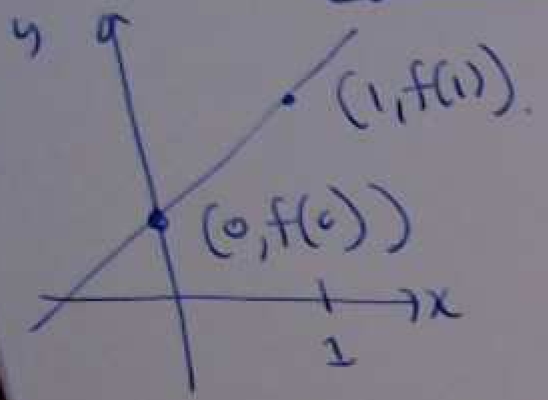
$$= \frac{\Delta y}{\Delta x}$$



$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \text{slope}$$

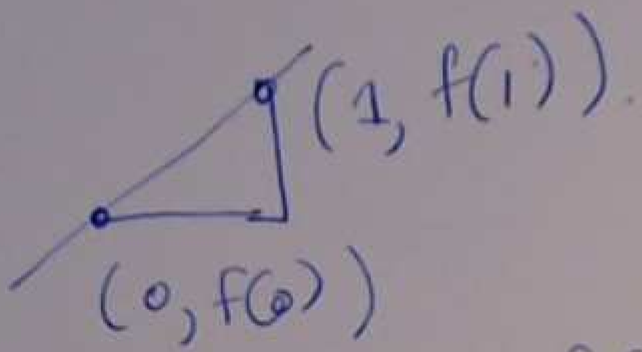
special case

$\boxed{y=mx+b}$



graph of f: $(x, f(x))$

$f(x) = \underbrace{m}_{\text{slope}}x + b$

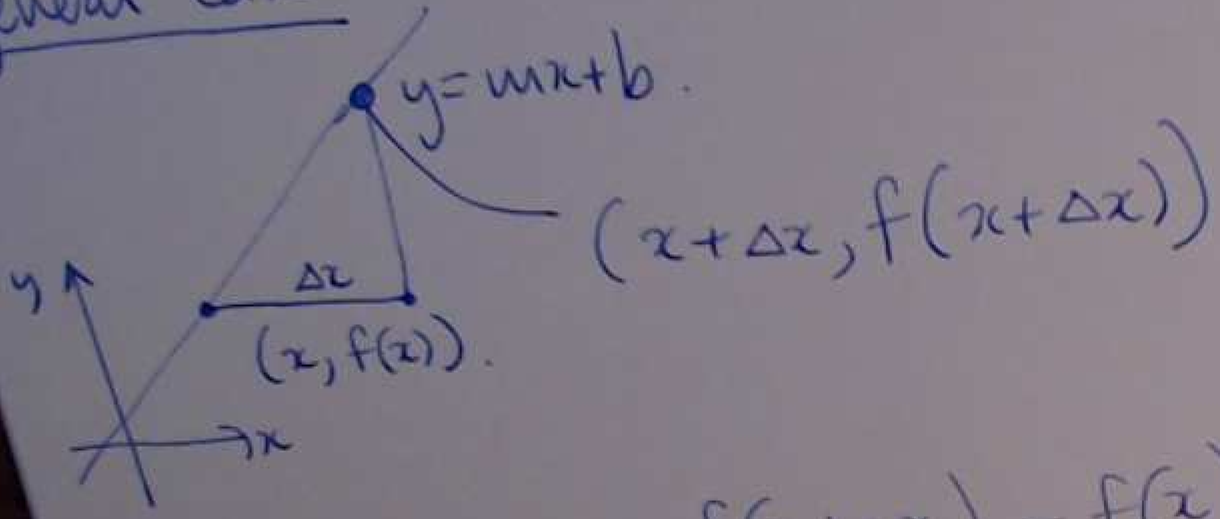


$$\frac{\Delta y}{\Delta x} = \frac{f(1) - f(0)}{1 - 0}$$

$$= \frac{m + b - b}{1}$$

$$= m \leftarrow \text{slope of line}$$

general case



useful fact

straight lines have constant slope everywhere.

slope $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\cancel{x}+\Delta x-\cancel{x}}$

$$= \frac{m(x+\Delta x)+b-(mx+b)}{\Delta x}$$

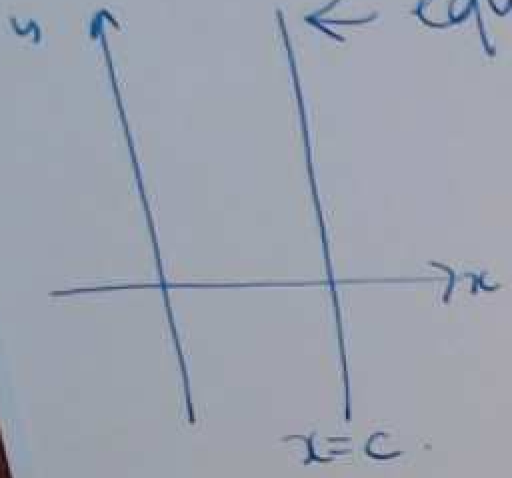
$$= \frac{\cancel{mx}+m\cancel{\Delta x}+\cancel{b}-\cancel{mx}-\cancel{b}}{\cancel{\Delta x}} = m$$

observations

- $|m|$ large, steep slope

  +ve -ve.

- $m = 0$ horizontal line
- $m > 0$ increasing (left to right).
- $m < 0$ decreasing
- vertical lines are not graphs of functions.



← equation of vertical line is $x=c$

equation vs function

relation between variables

e.g. $x = c$.

$x^2+y^2 = 1$.

$f : \mathbb{R} \to \mathbb{R}$

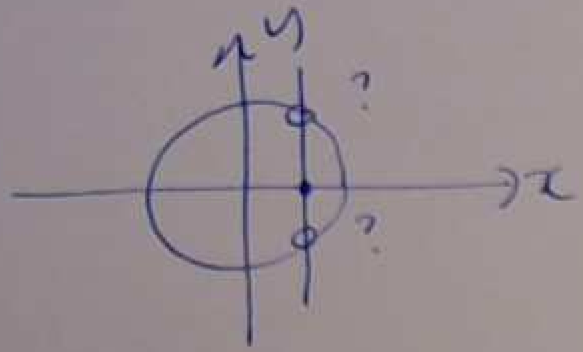
a map from one set to another set.

the graph of $f : \mathbb{R} \to \mathbb{R}$ is given by $\underbrace{y = f(x)}_{\text{equation}}$.

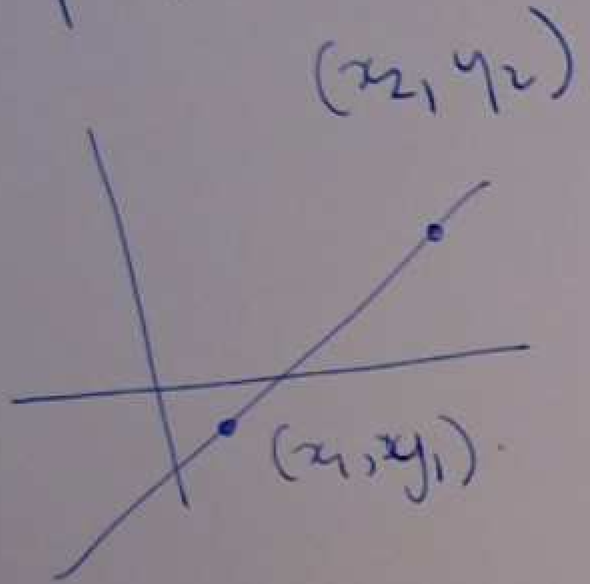
$\uparrow (x, f(x)) \subset \mathbb{R}^2$

<u>but</u> not every equation comes from a function.

eg. $x^2 + y^2 = 1$.



useful technique find equation of line through two points.



- find the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- find the line.

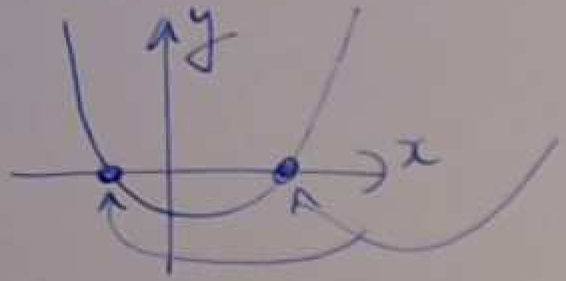
point-slope formula for a line. → $y - y_1 = m(x - x_1)$

Quadratic functions

given by $f(x) = ax^2 + bx + c$.

(a, b, c constants, don't depend on x).

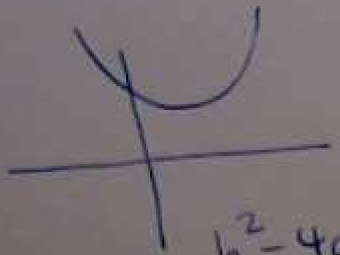
graphs are parabolas.



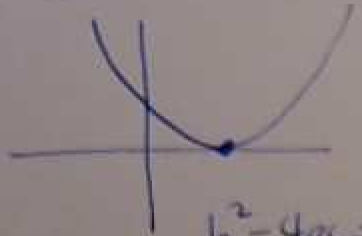
solutions to $f(x)=0$ are roots.

can solve $f(x)=0$

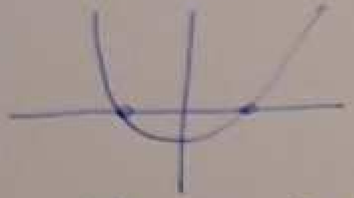
at most two, given by $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.



$b^2-4ac<0$.



$b^2-4ac=0$



$b^2-4ac>0$.

useful techniques

- factorization: $ax^2+bx+c = a(x-r_1)(x-r_2)$

  here r_1, r_2 are roots / solutions to $f(x)=0$

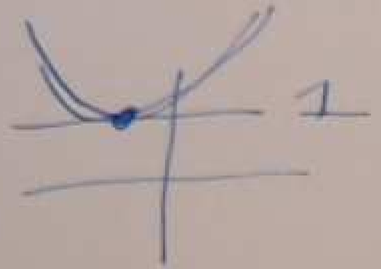
- complete the square: any quadratic expression can be written as $a(x+b)^2+c$.

$(x+b)^2 = x^2+2b+b^2$.

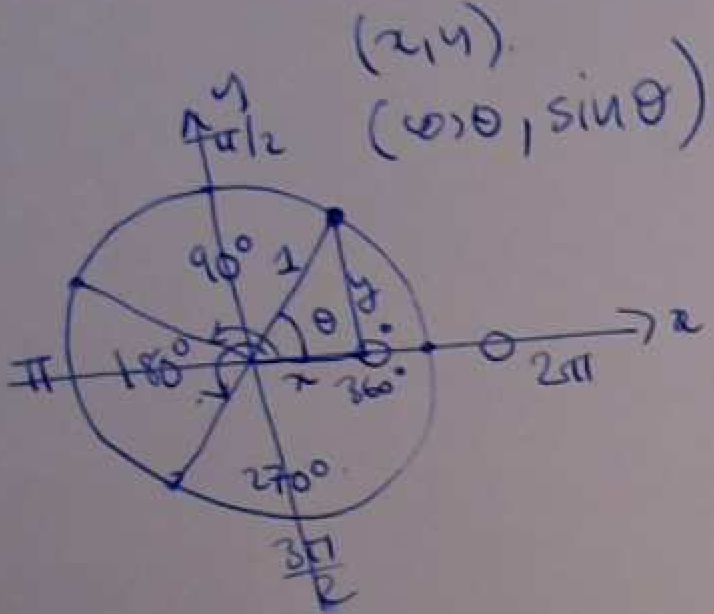
<u>example</u> x^2+2x+3.

$(x+1)^2+1$ ← complete the square.

$x^2+2x+2+1$

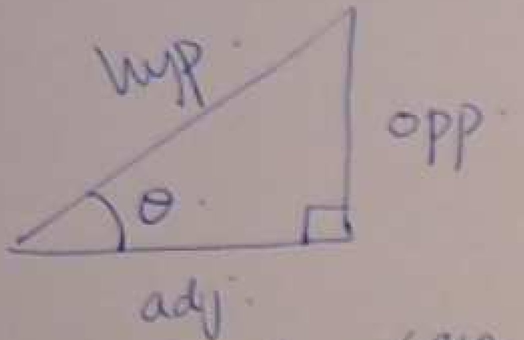


§1.4 Trig functions



angles: ~~degrees~~ vs $\boxed{\text{radians}}$

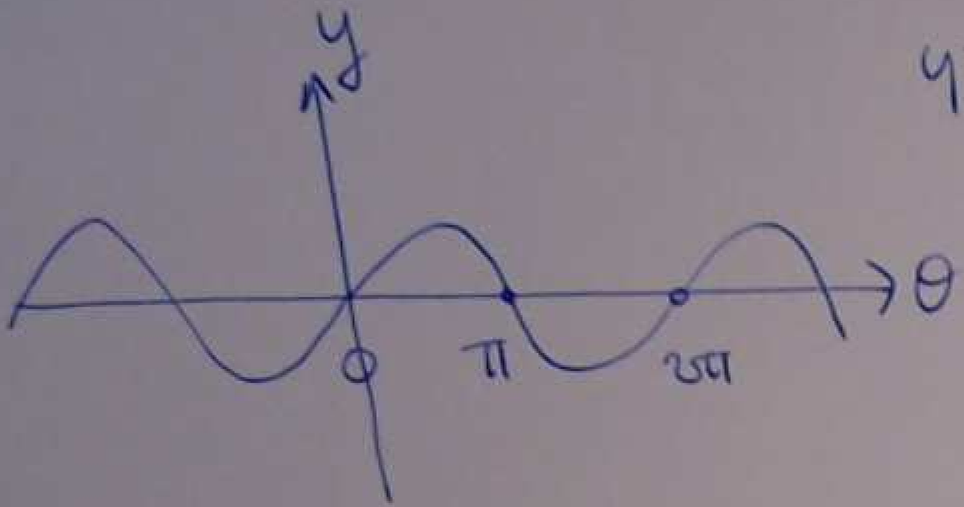
angle in radians = distance around the unit circle



• right angled triangles

$\sin\theta = \dfrac{opp}{hyp}$ $\qquad \cos\theta = \dfrac{adj}{hyp}$ } can extend to be defined for all $\theta \in \mathbb{R}$

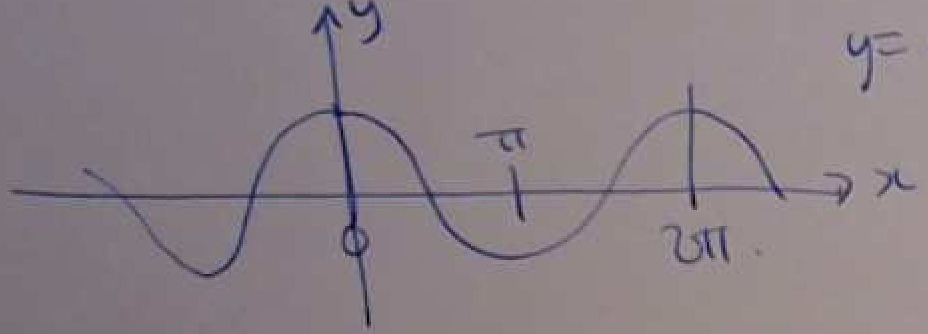
useful facts



$y = \sin(\theta)$

$$\sin(-\theta) = -\sin(\theta)$$

(odd function)



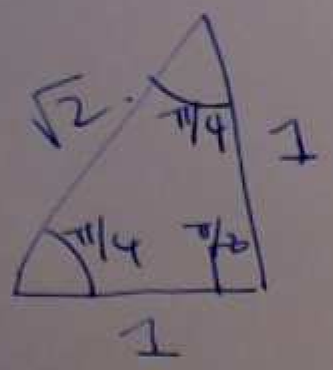
$y = \cos(x)$

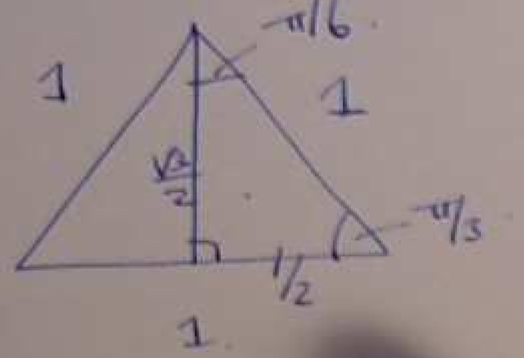
$$\cos(-x) = \cos(x)$$

(even function)

reflectional symmetry in the y-axis.

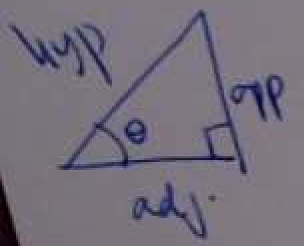
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$\sqrt{1}/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{4}/2$
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0





other trig functions

$$\tan(x) = \frac{\text{opp}}{\text{adj}} = \frac{\sin x}{\cos x}$$



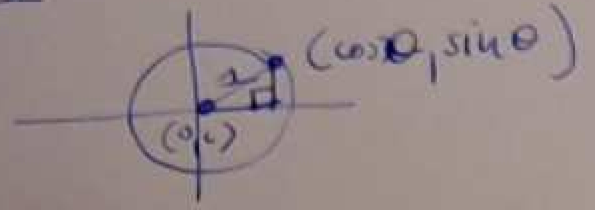
$$\frac{1}{\sin x} = \operatorname{cosec}(x).$$

$$\frac{1}{\cos x} = \sec(x)$$

$$\frac{1}{\tan x} = \cot(x).$$

Trig identities

Pythagorean identity $\qquad \sin^2 x + \cos^2 x = 1.$



Unpin Video Recording...

17

$$\sin^2 x + \cos^2 x = 1.$$

$$\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \longleftrightarrow 1 + \cot^2 x = \csc^2 x.$$

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \longleftrightarrow \tan^2 x + 1 = \sec^2 x.$$

<u>Double angle</u>: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$
$$= 2\cos^2\theta - 1.$$

Joseph Maher

Addition

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

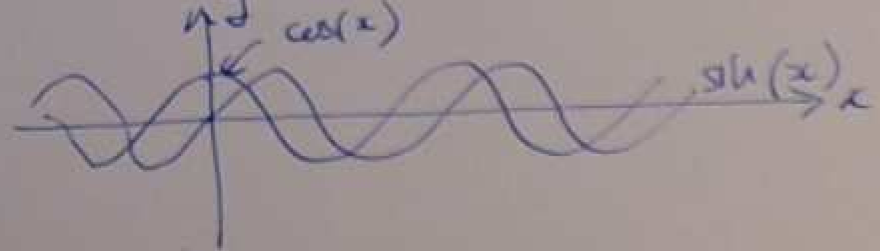
$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

$$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$

$$\| $$

$$\sin(2\theta)$$

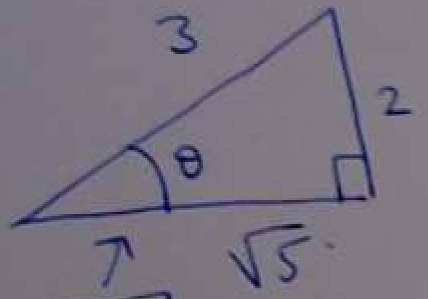
special case (shift).



$$\sin\left(x+\frac{\pi}{2}\right) = \cos(x).$$

$$\sin\left(x+\frac{\pi}{2}\right) = \sin(x)\overbrace{\cos\left(\frac{\pi}{2}\right)}^{0} + \cos(x)\overbrace{\sin\left(\frac{\pi}{2}\right)}^{1}.$$

Example suppose $\sin\theta = \frac{2}{3}$ find $\cos\theta$, $\tan\theta$, $\sin 2\theta$.



$\sqrt{3^2-2^2}$
$\sqrt{9-4}$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{5}}{3}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{2}{\sqrt{5}}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\,\frac{2}{3}\,\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$$

§2.5 Inverse functions

recall: $f: A \longrightarrow B$.

domain range

$x \longmapsto f(x)$.

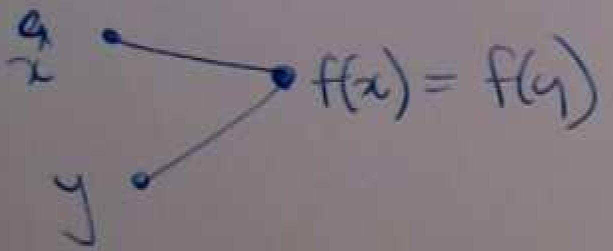
want: the inverse function should be the reverse of this.

$f^{-1}: B \longrightarrow A$

$f(x) \longmapsto x$

problem

the inverse is often not a function.



if $x \neq y$

but $f(x) = f(y)$ then

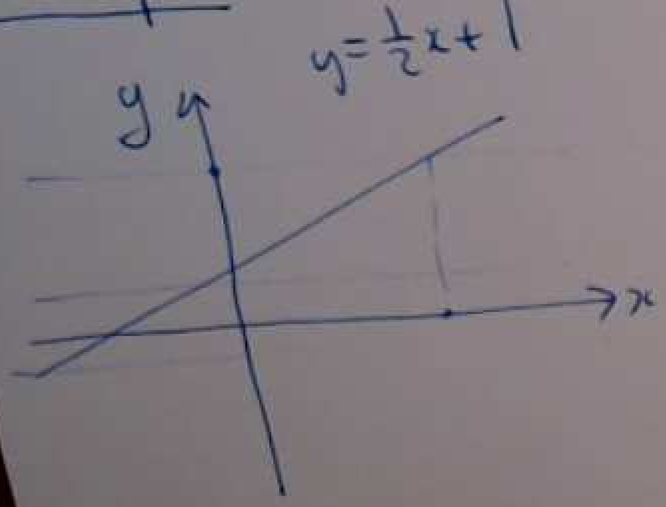
what is $f^{-1}(f(x))$?

Q: when does a function $f:\mathbb{R}\to\mathbb{R}$ have an inverse?

A: when it passes the horizontal line test (one-to-one / injective).

$\iff$ for each $c \in$ range, there is a unique $x \in$ domain s.t. $f(x) = c$.

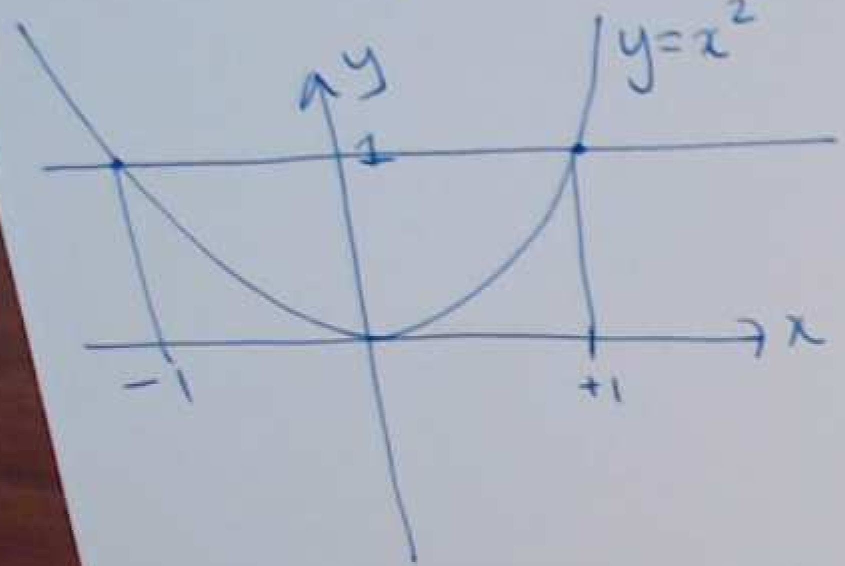
Example



Q: how do we find a formula for the inverse?

A: ① write down $y = f(x)$
② solve for x in terms of y
$x = g(y)$
③ $f^{-1}(x) = g(x)$
④ check!

bad example $f(x) = x^2$ ← $f: \mathbb{R} \longrightarrow \mathbb{R}$.



problem doesn't pass horizontal line test, no inverse function.

fix: restrict domain.

define new function

$$f: [0, \infty) \longrightarrow [0, \infty)$$
$$x \longmapsto x^2.$$

has an inverse.

$$f^{-1}(x) = \sqrt{x}$$

$f^{-1}: [0, \infty) \longrightarrow [0, \infty)$ $\quad x \longmapsto \sqrt{x}$.

how to draw the graph of the inverse function

$f: \mathbb{R} \longrightarrow \mathbb{R}$
$x \longmapsto f(x)$

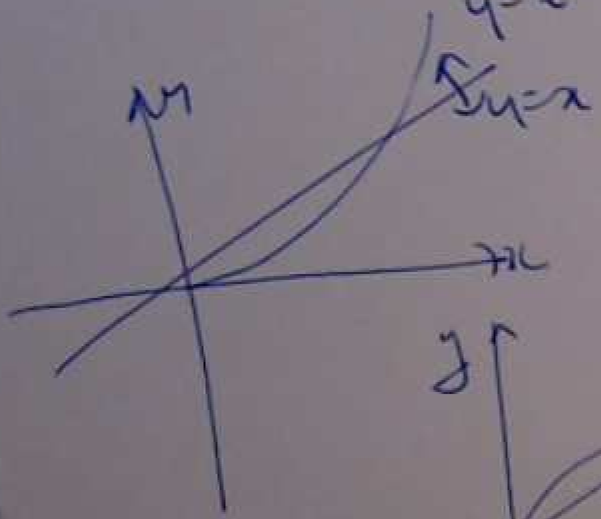
graph $(x, f(x))$

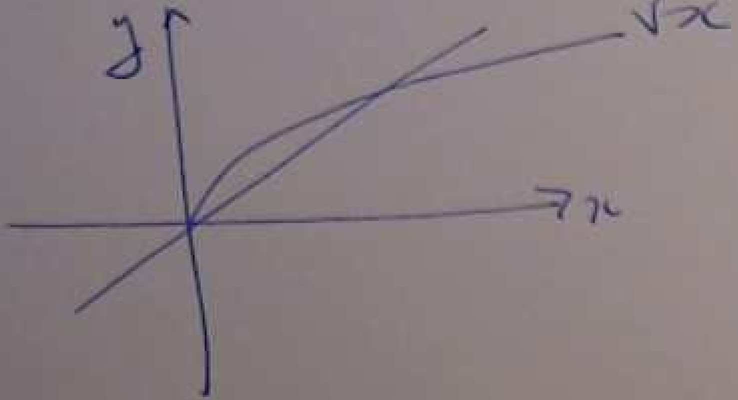
$f^{-1}: \mathbb{R} \longrightarrow \mathbb{R}$
$f(x) \longmapsto x$.

$(f(x), x)$.

$(x, y) \Leftrightarrow (y, x)$.

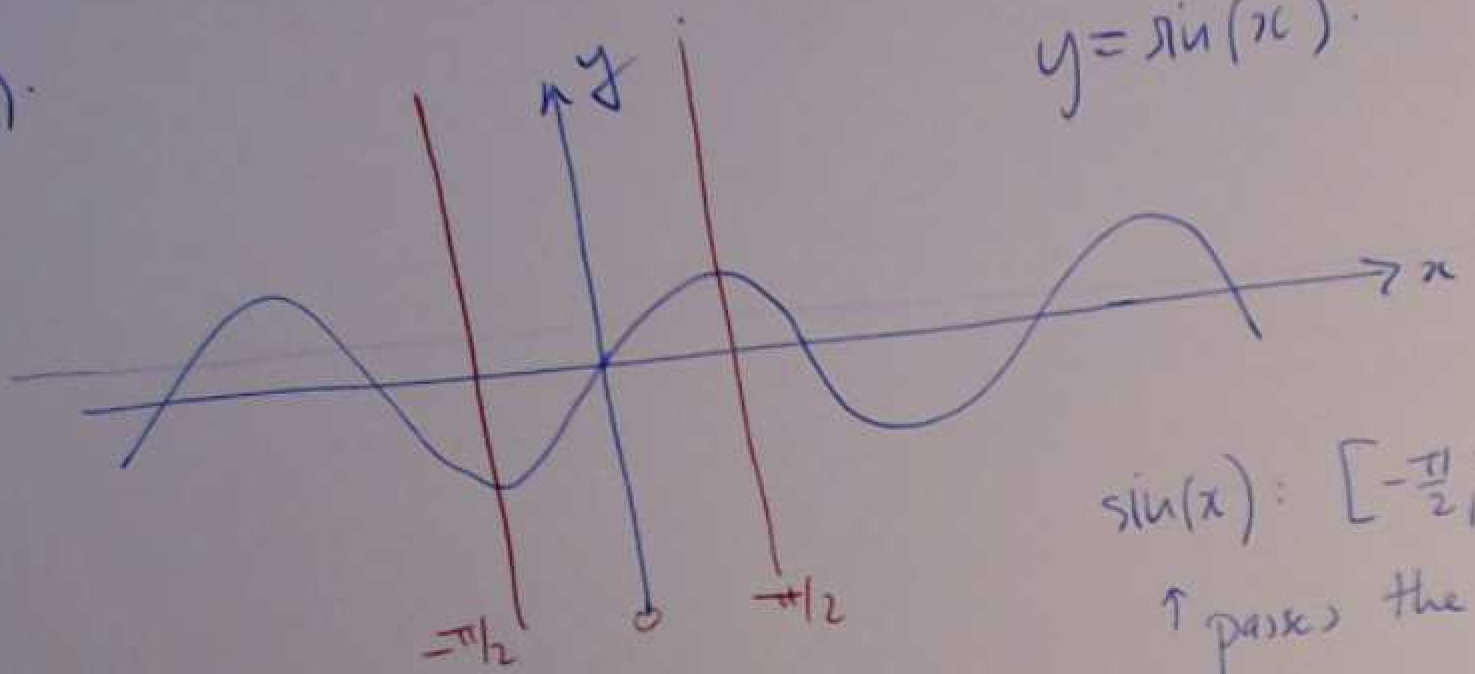
↑ reflection in $y = x$





sin(x).

$y = \sin(x)$.

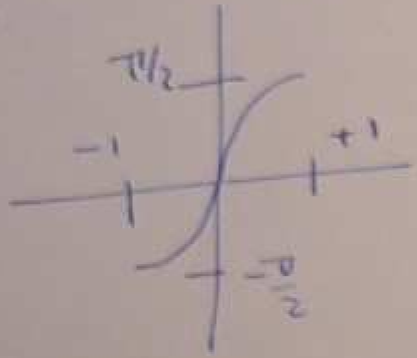


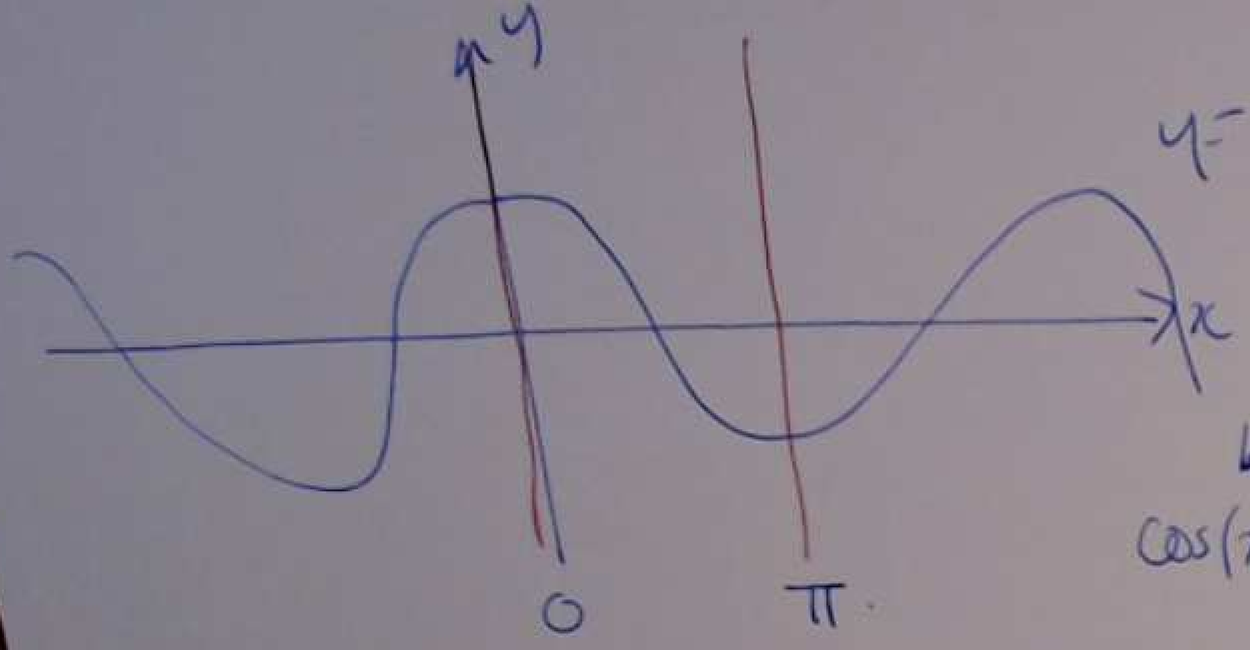
$\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$.

↑ passes the horizontal line test.

$\sin^{-1}(x) : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$.

arcsin(x)



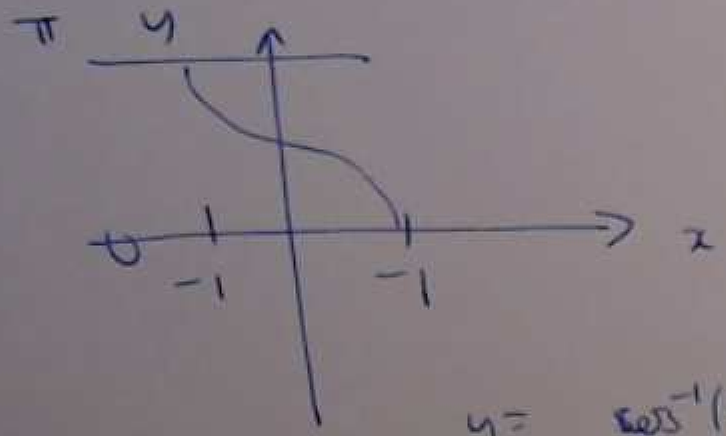


$y = \cos(x)$

restricted cos

$\cos(x) : [0, \pi] \to [-1, 1]$.

$\cos^{-1}(x) : [-1, 1] \to [0, \pi]$



$y = \cos^{-1}(x)$