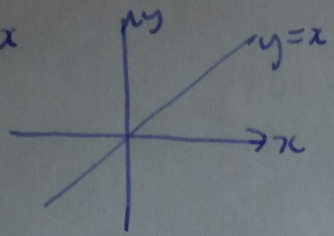
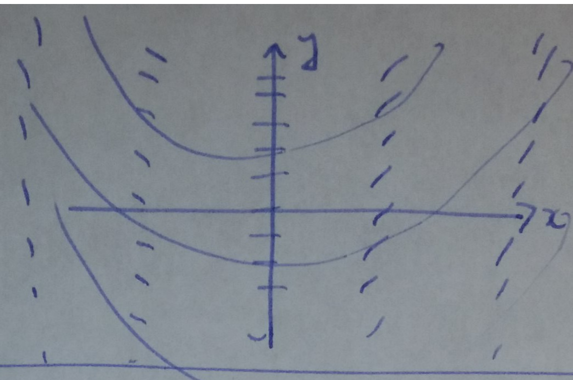
$f(x) = x$





$F(x) = \frac{1}{2}x^2 + c$

Example find the general antiderivative to $f(x) = \sin(4x)$

guess: $\frac{d}{dx}(\cos(4x)) = -\sin(4x) \cdot \frac{d}{dx}4$

so $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x)\right) = -\frac{1}{4} \cdot -\sin(4x) \cdot 4 = \sin(4x)$

so $F(x) = -\frac{1}{4}\cancel{\sin}\cos(4x) + c$

Notation indefinite integral

$\int f(x)\,dx = F(x) + c$ means: $F(x)+c$ is the general antiderivative for $f(x)$

Thm $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$

Proof $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + c\right) = \frac{1}{n+1}\,n+1\,x^n = x^n$ $\square$

Thm $\int \frac{1}{x}\,dx = \ln|x| + c$

Proof $(x>0)$ $\frac{d}{dx}(\ln(x) + c) = \frac{1}{x}$ $\square$.

Thm (sums and constant multiples)

$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$ $\qquad$ $\int cf(x)\,dx = c\int f(x)\,dx$

warning: no product / quotient / chain rule!

useful integrals: $\int \sin(x)\,dx = -\cos x + c$

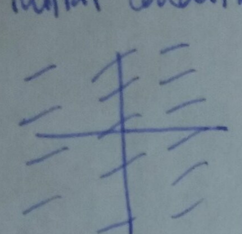
$\int \cos(x)\,dx = \sin x + c$

$\int e^x\,dx = e^x + c$

Example $\int x^2 + \frac{1}{x} + \sin(x)\,dx = \frac{1}{3}x^3 + \ln|x| - \cos(x) + C$

Alternate view

We can think of finding the indefinite integral as finding a function given its slope function, i.e. its derivative. This is an example of solving a differential equation $\frac{dy}{dx} = f(x)$. In general, there is a family of solutions $F(x)+c$, but if we know the value of the solution we want at $x=0$, (sometimes called an initial condition) then this gives a particular solution.



$\frac{dy}{dx} = 1$ general solution $F(x) = x + c$

initial condition $F(0) = 1$ then $F(x) = x + 1$.

Example objects falling freely under gravity, so acceleration $a(t) = -g$ (constant), velocity $v(t)$ has $v'(t) = a(t)$

$$\frac{dv}{dt} = -g$$

has general solution $v(t) = -gt + c$

if the velocity at time $t=0$ is v_0, then $v(0) = v_0 = c$, and so the particular solution is $v(t) = -gt + v_0$.

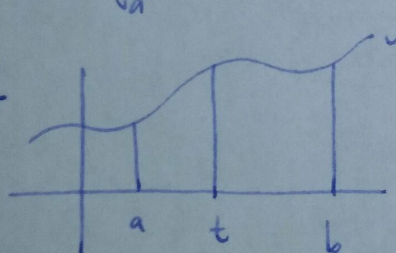
position: $x(t)$ has $\frac{dx}{dt} = v(t) = -gt + v_0$, general solution $x(t) = -\frac{1}{2}gt^2 + tv_0 + c$

if position at time $t=0$ is x_0, then $x(t) = -\frac{1}{2}gt^2 + tv_0 + x_0$.

§5.4 Fundamental theorem of calculus

Thm (FTC①) suppose $f(x)$ is cts on $[a,b]$ and $F(x)$ is an antiderivative for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$

intuition



consider $\int_a^t f(x)\,dx$

Q: what is the rate of change wrt to t?

$$\frac{d}{dt}\int_a^t f(x)\,dx = \lim_{h\to 0} \frac{\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx}{h} = \lim_{h\to 0} \frac{\int_t^{t+h} f(x)\,dx}{h}$$