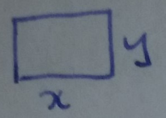
§4.6 Optimization

Example a piece of wire of length L is bent into a rectangle. What is the largest possible area?

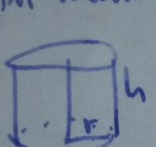


$$\left.\begin{array}{l} \text{area } A = xy \\ \text{length } L = 2x+2y \end{array}\right\} \quad y = \frac{L}{2} - x \qquad A = x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2$$

$A'(x) = \frac{L}{2} - 2x$ critical point $x = \frac{L}{4}$ (max)

so max area of $\frac{L^2}{16}$ occurs when $x = y = \frac{L}{4}$ (square).

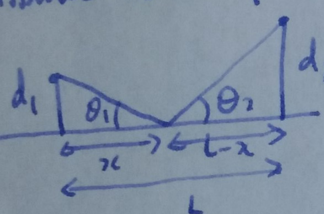
Example what shape of cylindrical can minimizes surface area, if you want total volume to be $1\ ft^3$.



$$\left.\begin{array}{l} V = \pi r^2 h = 1 \iff h = \frac{1}{\pi r^2} \\ A = 2\pi r^2 + 2\pi r h \end{array}\right\} \quad A = 2\pi r^2 + 2\pi r \frac{1}{2\pi r^2} = 2\pi r^2 + \frac{2}{r}$$

$A'(r) = 4\pi r - \frac{2}{r^2}$ solve $A'(r) = 0$: $r^3 = \frac{1}{2\pi}$ $r = \sqrt[3]{1/2\pi}$.

Example a ball bounces off a wall. Show that the path which minimizes distance has equal angle of incidence and reflection.



$f(x) = \text{distance} = \sqrt{d_1^2 + x^2} + \sqrt{d_2^2 + (L-x)^2}$

$f'(x) = \frac{1}{2}(d_1^2 + x^2)^{-1/2} \cdot 2x + \frac{1}{2}(d_2^2 + (L-x)^2)^{-1/2} \cdot 2(L-x) \cdot -1$.

$f'(x) = 0$: $\frac{x}{\sqrt{d_1^2 + x^2}} = \frac{L-x}{\sqrt{d_2^2 + (L-x)^2}} \iff \cos\theta_1 = \cos\theta_2 \iff \theta_1 = \theta_2$.

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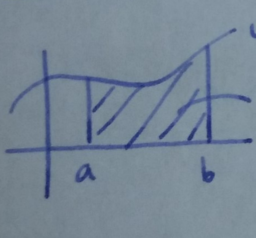
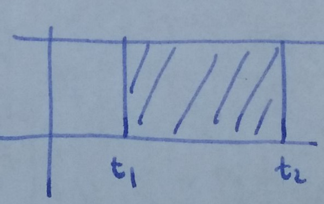
§5.1 Approximating area

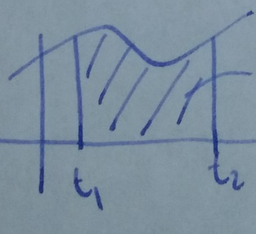


Example

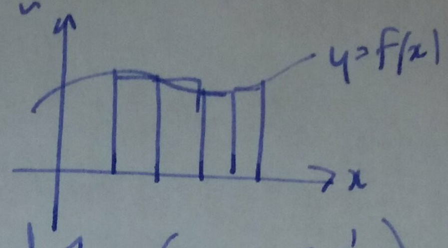


distance travelled = velocity × time
= area under the graph

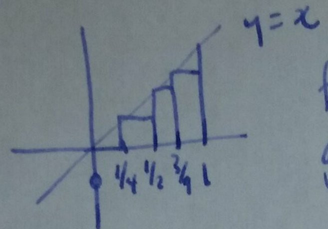
non-constant velocity:



distance travelled = area under the graph finding the area: approximate by rectangles



example



find area under graph of $y=x$ between 0 and 1 (answer $=\frac{1}{2}$).

approximate with 4 rectangles: area $\approx$ sum of $\underbrace{\text{area of rectangles}}_{\text{width}\times\text{height}}$

$$= \frac{1}{4}f(0)+\frac{1}{4}f(\tfrac{1}{4})+\frac{1}{4}f(\tfrac{1}{2})+\frac{1}{4}f(\tfrac{3}{4}) = \sum_{i=0}^{3} f(\tfrac{i}{4})\cdot\frac{1}{4}.$$

$$= \frac{1}{4}\left(f(0)+f(\tfrac{1}{4})+f(\tfrac{2}{4})+f(\tfrac{3}{4})\right).$$

$$= \frac{1}{4}\left(0+\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right) = \frac{1}{4}\cdot\frac{6}{4} = \frac{3}{8} = 0.375.$$

approximate with n rectangles

$$f(0)\frac{1}{n}+f(\tfrac{1}{n})\frac{1}{n}+f(\tfrac{2}{n})\frac{1}{n}+\cdots+f(\tfrac{n-1}{n})\frac{1}{n} = \sum_{i=0}^{n-1}\frac{1}{n}f(\tfrac{i}{n})$$

$$= \frac{1}{n}\left(f(0)+f(\tfrac{1}{n})+\cdots+f(\tfrac{n-1}{n})\right)$$

$$= \frac{1}{n}\left(0+\frac{1}{n}+\frac{2}{n}+\cdots+\frac{n-1}{n}\right) = \sum_{i=0}^{n-1}\frac{i}{n}\cdot\frac{1}{n}$$

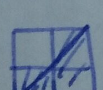
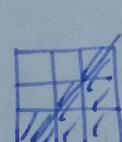
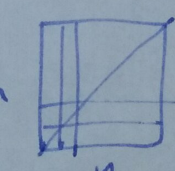
$$= \frac{1}{n^2}\left(1+2+3+\cdots+n-1\right) = \frac{1}{n^2}\sum_{i=0}^{n-1} i$$

claim $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$

Proof ① induction: assume true for k: $S_k = 1+2+\cdots+k = \frac{1}{2}k(k+1)$

$$S_{k+1} = \underbrace{1+2+\cdots+k}_{S_k}+k+1 = S_k+k+1 = \frac{1}{2}k(k+1)+k+1 = (k+1)(\tfrac{1}{2}k+1) = \frac{1}{2}(k+1)(k+2) = S_{k+1}\ ✓.$$

finally: check $k=1$: $1 = \frac{1}{2}\cdot 1\cdot 2$ ✓.

② 2  2 $\quad \frac{1}{2}(2)^2+\frac{1}{2}\cdot 2$ $\quad$ 3  3 $\quad \frac{1}{2}(3)^2+\frac{1}{2}\cdot 3$ $\quad$ n  n $\quad \frac{1}{2}(n)^2+\frac{1}{2}\cdot n.$

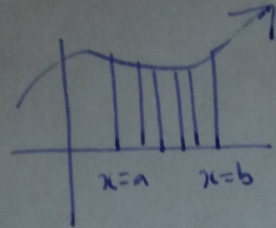
$$= \frac{1}{2}n(n+1)$$

③ $\underbrace{1+\underbrace{2+\cdots\cdots+n-1}_{n+1}+n}_{n+1}$ $\quad$ n even: $\frac{1}{2}n(n+1)$.

n odd: $\frac{n-1}{2}(n+1)+\frac{n+1}{2} = (n+1)\left(\frac{n-1}{2}+\frac{1}{2}\right) = \frac{1}{2}n(n+1)$ □

so approximate area is $\frac{1}{n^2}\left(1+2+\cdots+(n-1)\right) = \frac{1}{n^2}\frac{1}{2}(n-1)n = \frac{1}{2}\frac{n^2-n}{n^2} = \frac{1}{2}(1-\frac{1}{n}) \to \frac{1}{2}$ as $n\to\infty$.

Notation

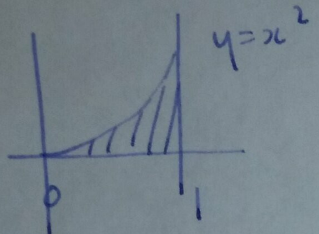


N rectangles of equal width then $\Delta x = \frac{b-a}{N}$

left endpoint rectangles $L_N = \sum_{i=0}^{N-1} f(a+i\Delta x)\Delta x$

right endpoint rectangles $R_N = \sum_{i=1}^{N} f(a+i\Delta x)\Delta x$

midpoint rectangles $M_N = \sum_{i=1}^{N} f(a+(i-\frac{1}{2})\Delta x)\Delta x$

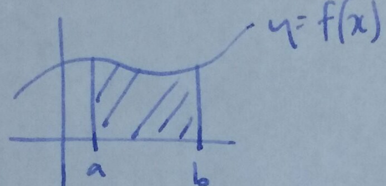


$$1+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1).$$

useful fact **Thm** If $f(x)$ is cts on $[a,b]$ then all of these approximations have the same limit as $N\to\infty$, which is the area under the curve.

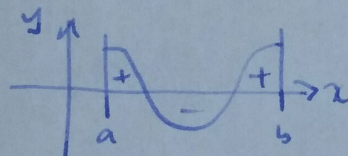
$$\text{area} = \lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N = \lim_{N\to\infty} M_N$$

§5.2 Definite integral

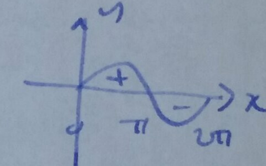


$\int_a^b f(x)\,dx$ = area under the curve between $x=a$ and $x=b$

note: signed area

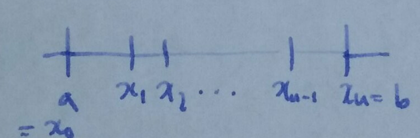


so $\int_0^{2\pi} \sin(x)\,dx = 0$



Formal definition Riemann sum $R(f,P,c)$

P = partition of $[a,b]$



widths $\Delta x_i = x_i - x_{i-1}$

c = choice of points $c_i \in [x_{i-1}, x_i]$ $\quad R(f,P,c) = \sum_{i=1}^{n} f(c_i)\Delta x_i$

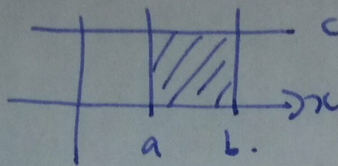
$\|P\| = \max \Delta x_i$

Defn $$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} R(f,P,c) = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

when this limit exists we say f is integrable over $[a,b]$.

useful properties

$$\int_a^b c\,dx = c(b-a)$$

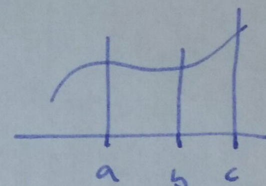


$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

0-length interval $\int_a^a f(x)\,dx = 0$



adjacent intervals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

comparisons: if $f(x) \le g(x)$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

§5.3 Antiderivatives

Defn A function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$

Example if $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$ is an antiderivative

check: $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$ ✓

Note: $F(x) = \frac{1}{3}x^3 + 4$ is also an antiderivative, $\frac{d}{dx}\left(\frac{1}{3}x^3+4\right) = x^2$

General antiderivative

Thm Let $F(x)$ be an antiderivative for $f(x)$, then any other antiderivative has the form $F(x)+c$ for some constant $c \in \mathbb{R}$.

Proof suppose $F(x)$ and $G(x)$ are antiderivatives for $f(x)$, then $F'(x) = f(x)$ and $G'(x) = f(x)$, so $(F(x)-G(x))' = f(x)-f(x) = 0$, so $F(x)-G(x)$ is the constant function. □.

Picture $f(x)$ gives the slope function for $F(x)$.

Examples

$f(x) = c$