Def^n^ $x=c$ is a vertical asymptote if $\lim_{x\to c^\pm} f(x) = \pm\infty$

$y=c$ is a horizontal asymptote if $\lim_{x\to\pm\infty} f(x) = c$

observation: rational functions $\frac{p(x)}{q(x)}$ have horizontal asymptotes if $\deg p \leq \deg q$

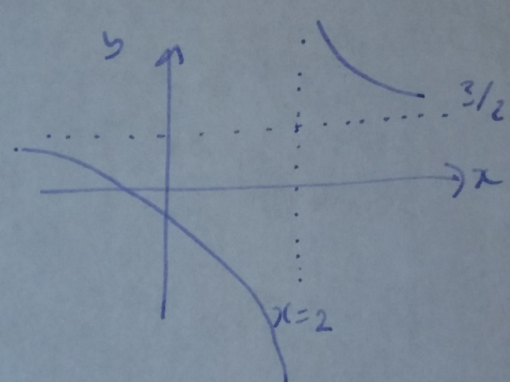
Example $f(x) = \frac{x^2+x+1}{3x^2+2} \sim \frac{x^2}{3x^2} \to \frac{1}{3}$ as $x \to \infty$

Example sketch graph of $f(x) = \frac{3x+2}{2x-4}$

① find vertical asymptotes $\leftrightarrow$ denominator zero : $2x-4=0$, $x=2$.

② find $f'(x) = \frac{(2x-4).3-(3x+2).2}{(2x-4)^2} = \frac{-16}{(2x-4)^2} = \frac{-4}{(x-2)^2}$ ← always −ve (decreasing)

no critical points except vertical asymptote at $x=2$.

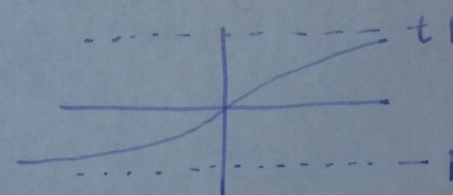


③ $f''(x) = \frac{8}{(x-2)^3}$ +ve $x>2$ concave up, −ve $x<2$ concave down.

④ horizontal asymptotes $\frac{3x+2}{2x-4} \sim \frac{3x}{2x} \sim \frac{3}{2}$

behaviour near asymptote:

$3x+2$	−	+	+
$2x-4$	−	−	+
	$-2/3$	2	
$f(x)$	+	−	+



Example sketch graph of $f(x) = \frac{x}{\sqrt{x^2+1}} = x(x^2+1)^{-1/2}$

① vertical asymptotes : none

② $f'(x) = \frac{\sqrt{x^2+1}\cdot 1 - x\frac{1}{2}(x^2+1)^{-1/2}}{(x^2+1)} = \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = (x^2+1)^{-3/2} > 0$ f always increasing, no critical points.

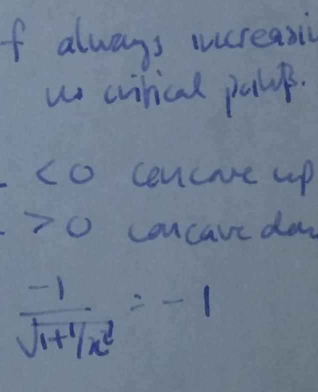
③ $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2}\cdot 2x$. point of inflection at $x=0$ +ve $x<0$ concave up, −ve $x>0$ concave down.

④ horizontal asymptotes:

$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{1+1/x^2}} = 1$ $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{-x}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{-1}{\sqrt{1+1/x^2}} = -1$

Example $f(x) = \frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}$ ③ $f''(x) = \frac{6x^2+2}{(x^2-1)^3}$

etc...



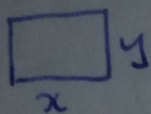
① vertical asymptotes at $x=\pm1$

② $f'(x) = -(x^2-1)^{-2}\cdot 2x$ critical pt at $x=0$

+2

§4.6 Optimization

<u>Example</u> a piece of wire of length L is bent into a rectangle. What is the largest possible area?

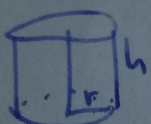


$$\left.\begin{array}{l}\text{area } A = xy \\ \text{length } L = 2x+2y\end{array}\right\} \quad y = \frac{L}{2} - x \qquad A = x\left(\frac{L}{2}-x\right) = \frac{L}{2}x - x^2$$

$A'(x) = \frac{L}{2} - 2x$ critical point $x = \frac{L}{4}$ (max)

so max area of $\frac{L^2}{16}$ occurs when $x = y = \frac{L}{4}$ (square).

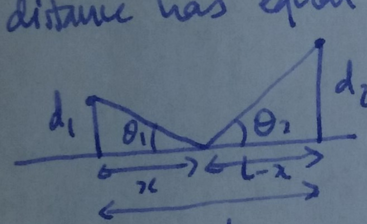
<u>Example</u> what shape of cylindrical can minimizes surface area, if you want total volume to be $1\,ft^3$.



$$\left.\begin{array}{l} V = \pi r^2 h = 1 \iff h = \frac{1}{\pi r^2} \\ A = 2\pi r^2 + 2\pi r h \end{array}\right\} \quad A = 2\pi r^2 + 2\pi r \frac{1}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$$

$A'(r) = 4\pi r - \frac{2}{r^2}$ solve $A'(r) = 0$: $r^3 = \frac{1}{2\pi}$ $r = \sqrt[3]{1/2\pi}$.

<u>Example</u> a ball bounces off a wall. Show that the path which minimizes distance has equal angle of incidence and reflection.

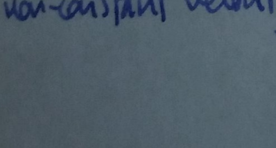


$f(x) = \text{distance} = \sqrt{d_1^2 + x^2} + \sqrt{d_2^2 + (L-x)^2}$

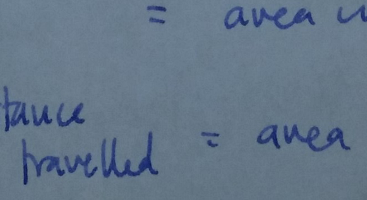
$f'(x) = \frac{1}{2}(d_1^2 + x^2)^{-1/2} \cdot 2x + \frac{1}{2}\left(d_2^2 + (L-x)^2\right)^{-1/2} \cdot 2(L-x) \cdot -1$.

$f'(x) = 0$: $\frac{x}{\sqrt{d_1^2 + x^2}} = \frac{L-x}{\sqrt{d_2^2 + (L-x)^2}} \iff \cos\theta_1 = \cos\theta_2 \iff \theta_1 = \theta_2$.

§5.1 Approximating area



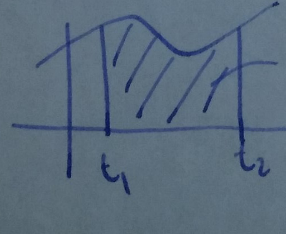
<u>Example</u>



constant velocity

distance travelled = velocity × time
= area under the graph

non-constant velocity:



distance travelled = area under the graph