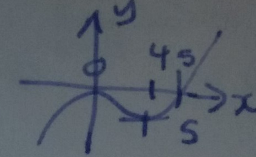


Example $f(x) = x^5 - 5x^4 = x^4(x-5)$.

$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$

$f''(x) = 20x^3 - 60x^2$

critical points: $f'(x) = 0 \Rightarrow x = 0, 4$.

2nd derivative test: $x=0$ $f''(0) = 0$ no information

$x=4$ $f''(4) = 320 > 0 \Rightarrow$ local min

at $x=0$ use first derivative test:

	0
$(x-4)$	$-$: $-$
x^3	$-$: $+$
$f'(x)$	$+$: $-$

$\Rightarrow$ local max.

§4.5 L'Hôpital's rule

Thm suppose $f(x)$ and $g(x)$ are differentiable and $f(a) = g(a) = 0$ (or $f(a) = g(a) = \pm\infty$).

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, if this limit exists.

warning: ① this is not the quotient rule!

② $\frac{f(a)}{g(a)}$ must be indeterminate form $\frac{0}{0}, \frac{\infty}{\infty}$ etc, can't use this on $\lim_{x\to 0} \frac{1}{x}$

Examples ① $\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{3x^2}{1} = 3$

② $\lim_{x\to \frac{\pi}{2}} \frac{\cos^2 x}{\sin x - 1} = \lim_{x\to\frac{\pi}{2}} \frac{-2\cos x \sin x}{\cos x} = \lim_{x\to\pi/2} -2\sin x = -2$

---

③ $\lim_{x\to 0+} x\ln(x) = \lim_{x\to 0+} \frac{\ln(x)}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x = 0$

④ $\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{e^x - 1}{\sin x} = \lim_{x\to 0} \frac{e^x}{\cos x} = -e$

⑤ $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = 0.$

6 $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln(x)}$ note e^x cts so $= e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1$

Comparing growth rates of functions

Q: which grows faster $(\ln(x))^2$ or $\sqrt{x}$?

$$\lim_{x\to\infty} \frac{\sqrt{x}}{(\ln(x))^2} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{2\ln(x)\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln(x)} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{4/x} = \lim_{x\to\infty} \frac{\sqrt{x}}{8} = \infty$$

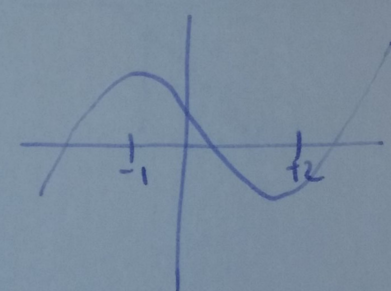
so $\sqrt{x}$ grows faster.

Thm e^x grows faster than any polynomial.

Proof $\lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{nx^{n-1}} = \cdots = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$.

§4.6 Graph sketching and asymptotes

Example sketch graph of $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3 = f(x)$



helpful info: · critical points $f'(x) = x^2 - x - 2$
· sign of $f'(x)$
· sign of $f''(x)$ $f''(x) = 2x - 1$

critical points at $x = -1, +2$

$f''(-1) = -3 < 0$ max

$f''(2) = 3 > 0$ min

Example $f(x) = (4x - x^2)e^x$ critical points $x = 1 \pm \sqrt{5}$

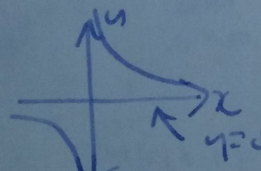
$f'(x) = (4 + 2x - x^2)e^x$

$f''(x) = (6 - x^2)e^x$

$1+\sqrt{5}$ local max $1-\sqrt{5}$ local min.

Asymptotes

Example $y = \frac{1}{x}$



<u>Def^{n}</u> $x=c$ is a <u>vertical asymptote</u> if $\lim_{x\to c^{\pm}} f(x) = \pm\infty$

$y=c$ is a <u>horizontal asymptote</u> if $\lim_{x\to\pm\infty} f(x) = c$

<u>observation</u>: rational function $\frac{p(x)}{q(x)}$ have horizontal asymptotes if $\deg p \leq \deg q$

<u>Example</u> $f(x) = \frac{x^2+x+1}{3x^2+2} \sim \frac{x^2}{3x^2} \to \frac{1}{3}$ as $x \to \infty$

<u>Example</u> sketch graph of $f(x) = \frac{3x+2}{2x-4}$

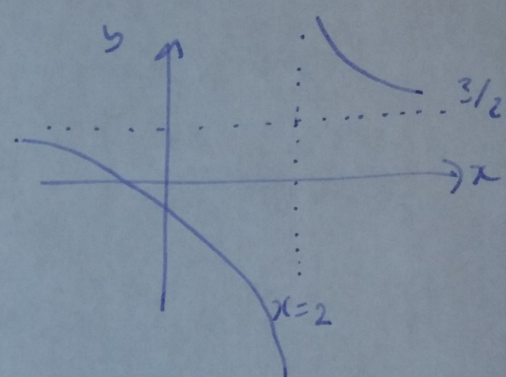
① find vertical asymptotes $\leftrightarrow$ denominator zero : $2x-4=0$, $x=2$.

② find $f'(x) = \frac{(2x-4).3-(3x+2).2}{(2x-4)^2} = \frac{-16}{(2x-4)^2} = \frac{-4}{(x-2)^2}$ $\leftarrow$ always $-$ve (decreasing)

no critical points except vertical asymptote at $x=2$.

③ $f''(x) = \frac{8}{(x-2)^3}$ $+$ve $x>2$ concave up

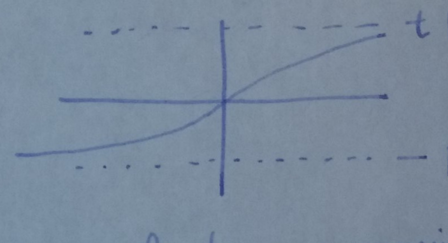
$-$ve $x<2$ concave down.



④ horizontal asymptotes $\frac{3x+2}{2x-4} \sim \frac{3x}{2x} \sim \frac{3}{2}$

behaviour near asymptote:

$3x+2$	$-$	$+$	$+$
$2x-4$	$-$	$-$	$+$
	$-2/3$		2
$f(x)$	$+$	$-$	$+$



<u>Example</u> sketch graph of $f(x) = \frac{x}{\sqrt{x^2+1}} = x(x^2+1)^{-1/2}$

① vertical asymptotes : none

② $f'(x) = \frac{\sqrt{x^2+1}\cdot 1 - x\frac{1}{2}(x^2+1)^{-1/2}}{(x^2+1)} = \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = (x^2+1)^{-3/2} > 0$ f always increasing, no critical points.

③ $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2}\cdot 2x$. point of inflection at $x=0$ $+$ve $x<0$ concave up

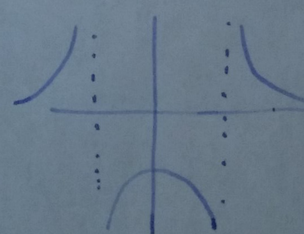
$-$ve $x>0$ concave down.

④ horizontal asymptotes:

$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{1+1/x^2}} = 1$ $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{-x}{\sqrt{x^2+1}} = \lim_{x\to+\infty} \frac{-1}{\sqrt{1+1/x^2}} = -1$

<u>Example</u> $f(x) = \frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}$ ③ $f''(x) = \frac{6x^2+2}{(x^2-1)^3}$

etc...



① vertical asymptotes at $x = \pm 1$

② $f'(x) = -(x^2-1)^{-2}\cdot 2x$ critical pt at $x=0$