Thm __First derivative test__ If $f(x)$ is differentiable, and $f'(c)=0$

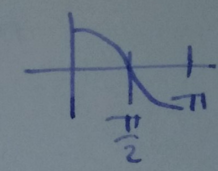
then if $f'(x)$ changes from +ve to −ve at c $\Rightarrow$ c local max

−ve to +ve $\Rightarrow$ c local min

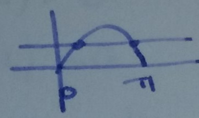
__Example__ classify critical points of $f(x) = \cos^2 x + \sin x$ on $[0,\pi]$

find critical points: $f'(x) = -2\cos(x)\sin x + \cos x$

solve $f'(x)=0$ $\quad \cos(x)(1-2\sin x)=0$ $\quad \cos(x)=0$:



$1-\sin 2x = 0 \iff \sin 2x = \frac{1}{2}$



$x = \frac{\pi}{6}, \frac{5\pi}{6}$

so critical points are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

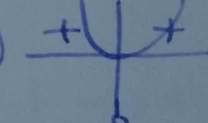
find sign of $f'(x)$:

		$\frac{\pi}{6}$		$\frac{\pi}{2}$		$\frac{5\pi}{6}$	
$\cos(x)$	+		+		−		−
$1-2\sin(x)$	+		−		−		+
$f'(x)$	+		−		+		−
		max		min		max.	

__Example__ critical point __not__ local max or min.

$f(x) = x^3$ $\quad f'(x) = 3x^2$ $\quad f'(x)=0$ $\quad 3x^2=0$ $\quad x=0$

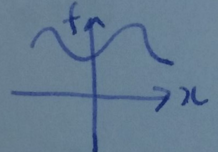
$f'(x)$



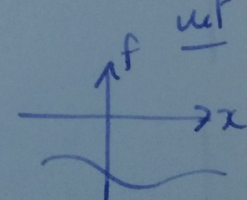
__not__ local max or min

__§4.4 Second derivative test__

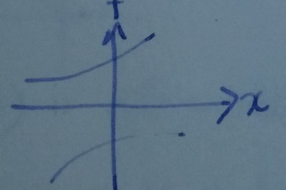
__recall__ $f(x)>0$ f positive



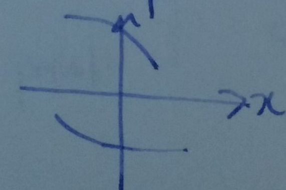
$f(x)<0$ f negative



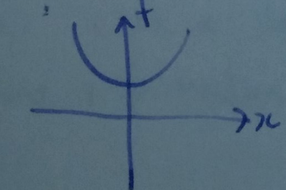
$f'(x)>0$ f increasing



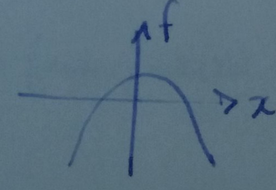
$f'(x)<0$ f decreasing



$f''(x)>0$ f concave up



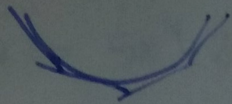
$f''(x)<0$ concave down

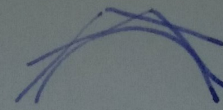


__mnemonic__

down up.

Q: how do slopes change?





concave down $\iff$ slopes decreasing $\iff f''(x) < 0$

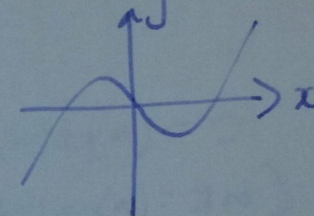
concave up $\iff$ slopes increasing $\iff f''(x) > 0$

<u>Defn</u> Let $f(x)$ be differentiable on an interval (a,b) then

$f(x)$ is concave up $\iff f''(x) > 0$
$f(x)$ is concave down $\iff f''(x) < 0$

<u>Defn</u> A <u>point of inflection</u> is where the graph changes from concave up to concave down, or vice versa.

<u>Note</u>: point of inflection $\Rightarrow f''(x) = 0$, $\not\Leftarrow$



<u>Example</u> $f(x) = x^3 - x = x(x-1)(x+1)$

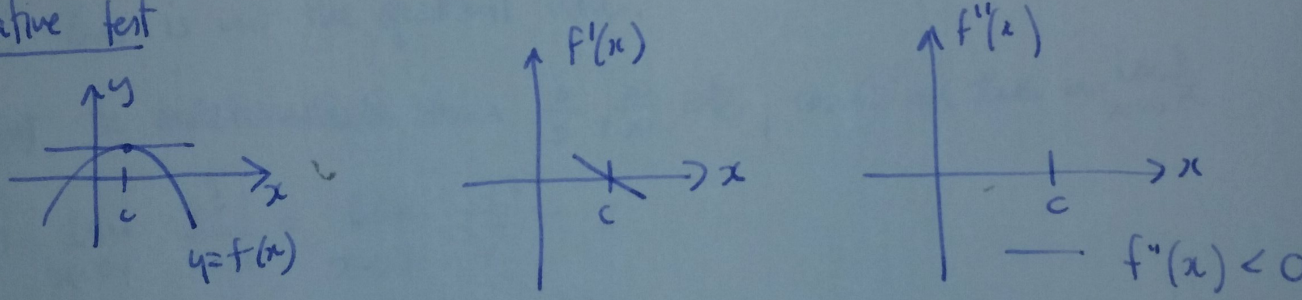
$f'(x) = 3x^2 - 1$

$f''(x) = 6x$

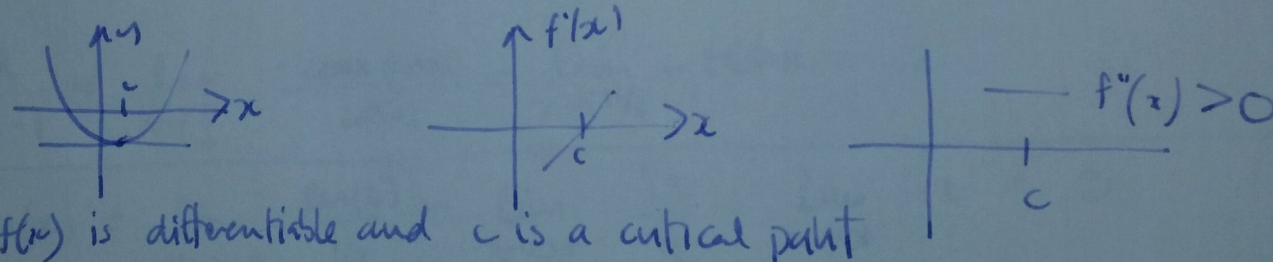
$f''(x) > 0$ for $x > 0$
$f''(x) < 0$ for $x < 0$

<u>Second derivative test</u>

local max:

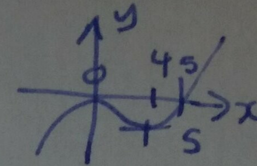


local min:



<u>Thm</u> suppose $f(x)$ is differentiable and c is a critical point

If $f''(c) > 0 \Rightarrow c$ is a local min
$f''(c) < 0 \Rightarrow c$ is a local max
$f''(c) = 0$ no information! (may be local max/min/neither)



Example $f(x) = x^5 - 5x^4 = x^4(x-5)$.

$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$

$f''(x) = 20x^3 - 60x^2$

critical points: $f'(x) = 0 \Rightarrow x = 0, 4$.

2nd derivative test: $x=0$ $f''(0) = 0$ no information

$x=4$ $f''(4) = 320 > 0 \Rightarrow$ local min

at $x=0$ use first derivative test:

		0	
$(x-4)$	$-$	:	$-$
x^3	$-$	:	$+$
$f'(x)$	$+$	0	$-$

$\Rightarrow$ local max.

§4.5 L'Hôpital's rule

Thm suppose $f(x)$ and $g(x)$ are differentiable and $f(a) = g(a) = 0$ (or $f(a) = g(a) = \pm\infty$).

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, if this limit exists.

warning: ① this is not the quotient rule!

② $\frac{f(a)}{g(a)}$ must be indeterminate form $\frac{0}{0}, \frac{\infty}{\infty}$ etc, can't use this on $\lim_{x\to 0} \frac{1}{x}$

Examples ① $\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{3x^2}{1} = 3$

② $\lim_{x\to \frac{\pi}{2}} \frac{\cos^2 x}{\sin x - 1} = \lim_{x\to\frac{\pi}{2}} \frac{-2\cos x \sin x}{\cos x} = \lim_{x\to\pi/2} -2\sin x = -2$

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③ $\lim_{x\to 0+} x\ln(x) = \lim_{x\to 0+} \frac{\ln(x)}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x = 0$

④ $\lim_{x\to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x\to 0} \frac{e^x - 1}{\sin x} = \lim_{x\to 0} \frac{e^x}{\cos x} = -e$

⑤ $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x - \sin x}{x\sin x} = \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = 0.$