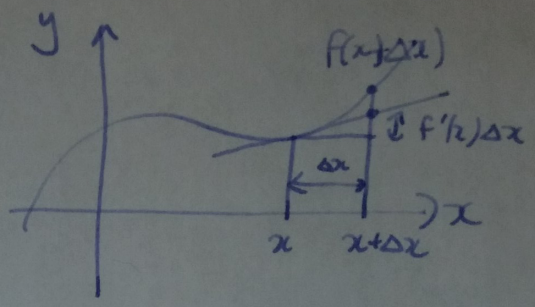
§4.1 Linear approximation

If $f(x)$ is differentiable at x, and Δx is small

then $f(x+\Delta x) \approx f(x) + \Delta x f'(x)$

so change in f, $\Delta f \approx f'(x)\Delta x$



Example estimate $\sqrt{103}$

$f(x) = \sqrt{x} = x^{1/2}$ $\quad f(100) = 10$ $\quad$ so $\Delta f \approx f'(x)\Delta x$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(100) = \frac{1}{20}$ $\quad \frac{1}{20} \cdot 3$

so $\sqrt{103} \approx 10 + \frac{3}{20} = 10.15$.

Example you make an 18" pizza. If your diameter is accurate to ± 0.4 in how much pizza do you gain or lose?

$A = \pi r^2$ $\quad 2r = D$ $\quad A = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{4}$ $\quad A'(D) = \frac{2\pi D}{4} = \frac{\pi D}{2}$

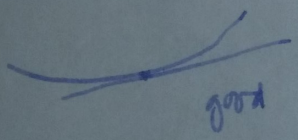
$\Delta A \approx A'(18)\Delta D = \frac{1}{2}\pi \cdot 18.0 \cdot 4 \approx 11 \text{ in}^2$.

Q is this good or bad? absolute error $= 11$

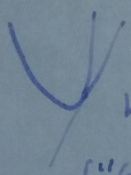
percentage error $= \left|\frac{\text{absolute error}}{\text{actual value}}\right| \times 100 = \frac{11}{\pi 18^2/4} \times 100. \approx 0.27\%$.

small.

Observation: when is the linear approximation a good approximation.



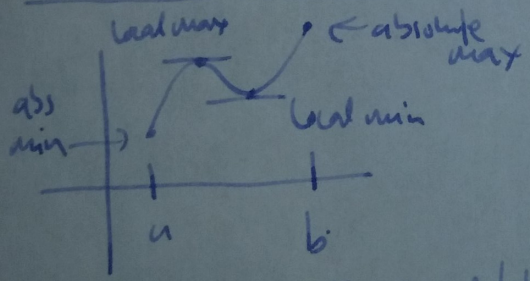
$f''(x)$ small



$f''(x)$ large.

§4.2 Extreme values

suppose $f(x)$ is defined on a closed interval $[a,b]$



Defn $f(c)$ is the absolute max if $f(c) \geq f(x)$ for all $x \in [a,b]$

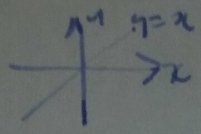
$f(c)$ is the absolute min if $f(c) \leq f(x)$ for all $x \in [a,b]$.

note: Q: where is the local/abs max/min ← want x-value

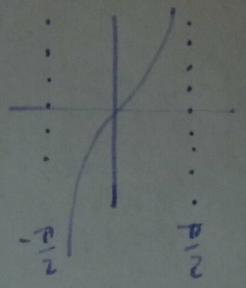
Q: what is the local/abs max/min ← want y-value

warning some functions do not have any max or min.

examples $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x$



$f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $x \mapsto \tan(x)$



Thm If $f(x)$ is continuous on a closed bounded interval $[a,b]$, then $f(x)$ has an absolute max and an absolute min.

Defn $f(x)$ has a local max at $x=c$ if there is a small interval containing c s.t. $f(c)$ is a maximum on this interval

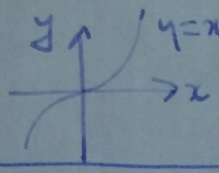
$f(x)$ has a local min at $x=c$ if there is a small interval containing c s.t. $f(c)$ is a minimum on this interval

Defn We say that c is a critical point if $f'(c)=0$ (or is undefined)

Thm If c is a local max or min, then c is a critical point.

warning: $f'(c)=0 \not\Rightarrow$ local max or min.

example $y=x^3$



$f'(x)=3x^2$ $f'(0)=0$ but $x=0$ not local max or min

How to find the absolute max or min of a differentiable function on a closed interval $[a,b]$: ① find critical points, evaluate function there.
② evaluate function at endpoints

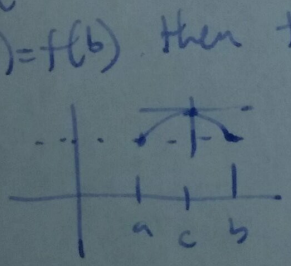
Example ① find abs max of $2x^3-15x^2+24x+7$ on $[0,3]$

② x^2-8 on $[1,4]$

③ $\cos(x)\sin(x)$ on $[0,4]$.

Thm (Rolle's Thm) If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) if $f(a)=f(b)$ then there is a $c \in (a,b)$ s.t. $f'(c)=0$.

Proof



- If there is a $c \in (a,b)$ which is a local max/min then $f'(c)=0$ ✓
- if no local max/min, then $f(x)=\text{const}=f(a)=f(b)$ so $f'(c)=0$ $\forall\, c \in (a,b)$ □.