Proof $y=\ln(x)$

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ $\square$.

Example ① $f(x) = \log_b(x) = \frac{\ln(x)}{\ln(b)}$ $\quad f'(x) = \frac{1}{x\ln(b)}$

② $f(x) = x\ln(x)$ $\quad f'(x) = \ln(x) + x\cdot\frac{1}{x} = \ln(x)+1$

Hyperbolic trig functions $\quad \sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$

$\frac{d}{dx}(\sinh(x)) = \cosh(x) \quad \frac{d}{dx}(\cosh(x)) = \sinh(x) \quad \boxed{\cosh^2 x - \sinh^2 x = 1}$

check: $(\sinh(x))' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh(x)$

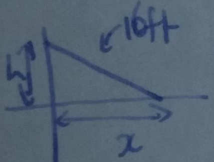
$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ so $\frac{d}{dx}(\tanh(x)) = \frac{1}{\cosh^2 x}$, etc.

inverse functions: $\frac{d}{dx}\left(\sinh^{-1}(x)\right) = \frac{1}{\sqrt{x^2+1}}$,

$\frac{d}{dx}\left(\cosh^{-1}(x)\right) = \frac{1}{\sqrt{x^2-1}} \quad \frac{d}{dx}\left(\tanh^{-1}(x)\right) = \frac{1}{1-x^2}$

§3.10 Related rates

Example falling ladder. $x(t)$ distance of foot of ladder from wall
$h(t)$ height of top of ladder against wall



Q: if $\frac{dx}{dt} = 3$ft/sec what is $\frac{dh}{dt}$?

spse at $t=0$, $x=5$ $\frac{dx}{dt} = 3$ft/se $\leftarrow$ specific to one time.

but $x^2+h^2 = 16^2$ $\leftarrow$ true for all time! also assume $\frac{dx}{dt} = 3$ft/se constant for all t.

$x(t) = 5+3t$

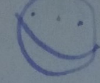
$(x(t))^2 + h(t)^2 = 16^2$ $\quad$ so $h(t) = \sqrt{16^2-(5+3t)^2}$

$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0 \quad 2\cdot 5\cdot 3 + 2\cdot\sqrt{16^2-5^2}\,\frac{dh}{dt} = 0$ $\quad (5+3t)$

so $\frac{dh}{dt} = -\frac{x(t)\cdot 3}{h(t)} = \frac{-3(3t+5)}{\sqrt{16^2+(3t+5)^2}}$,

hit ground at $5+3t=16$ $\quad t=\frac{11}{3}$ $\quad \frac{dh}{dt}(11/3) = \frac{-16}{\sqrt{16^2-11^2}} \to \infty$ as $t \to \frac{11}{3}$!

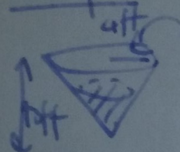
Example balloon: $V = \frac{4}{3}\pi r^3$



$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ $\quad \frac{dV}{dt} = 1\,ft^3/min$ $\quad r=1$ $\quad 1 = 4\pi \cdot 1 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{4\pi}$ ft/min.

Example filling a conical tank



$10ft^3/min$ $\quad$ Q: how fast is the water rising when $h=5ft$?

water in: $\frac{dV}{dt} = 10\,ft^3/min$

volume of cone: $V = \frac{1}{3}\pi h r^3$, need r in terms of h: $\frac{r}{h} = \frac{4}{10}$

i.e. $r = \frac{2}{5}h$, so $V = \frac{1}{3}\pi h\left(\frac{2}{5}h\right)^2 = \frac{4}{75}\pi h^3$

$\frac{dV}{dt} = \frac{12}{75}\pi h^2 \frac{dh}{dt}$, so when $h=5$: $\quad 10 = \frac{12}{75}\pi(5)^2\frac{dh}{dt}$

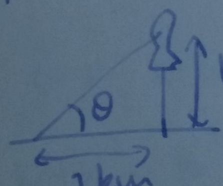
$\frac{dh}{dt} = \frac{10}{4\pi}$ ft/min.

advice ① give things names

② write down relations between things and use implicit differentiation.

③ plug in numbers if necessary

Example



if angle $= \frac{\pi}{3}$ and rate of change is $\frac{d\theta}{dt} = \frac{1}{2}$ rad/min

how fast is the rocket going?

$\frac{h}{1} = \tan\theta$ $\quad \frac{dh}{dt} = \sec^2\theta \frac{d\theta}{dt}$ $\quad \frac{dh}{dt} = \sec^2\left(\frac{\pi}{3}\right)\cdot\frac{1}{2}$

$\hat{=}$ 1km/min.