② $\sin^2(x) = f(g(x))$ where $f(x) = x^2 \quad f'(x) = 2x$

$g(x) = \sin(x) \quad g'(x) = \cos(x)$

so $(\sin^2(x))' = f'(g(x)).g'(x) = 2\sin(x)\cos(x)$.

③ $\sqrt{x^3+1}$ ok.

<u>Alternative notation</u> $f(g(x)) \longleftrightarrow f(u) \quad u = g(x)$

$$\frac{df}{dx} = f'(u)\frac{du}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}}$$

mnemonic: "cancel fractions"

<u>Examples</u> $\cos(x^2)$, $e^{\sqrt{x}}$, $\sin\left(\frac{\pi x}{180}\right)$, $\sqrt{x+\sqrt{x^3+1}}$.

<u>Proof</u> (of chain rule) [answer should be $f'(g(x))\,g'(x)$]

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

write this as: $\displaystyle\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$ ⊛.

set $k = g(x+h) - g(x)$, as g is cts $h \to 0 \Rightarrow k \to 0$

so $\displaystyle\lim_{k\to 0} \frac{f(g(x)+k) - f(g(x))}{k} = f'(g(x))$

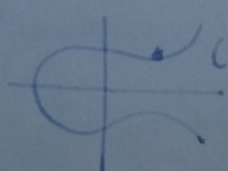
so ⊛ $= f'(g(x)).g'(x)$ as required □.

<u>More examples</u> $\dfrac{d}{dx}\left((g(x))^n\right) = n(g(x))^{n-1}.g'(x)$

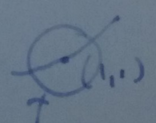
$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}.g'(x)$$

$$\frac{d}{dx}\left(f(ax+b)\right) = f'(ax+b).a.$$

<u>§3.8 Implicit differentiation</u>



locally



can think of $y(x)$

Suppose $y^4 + xy = x^3 - x + 2$

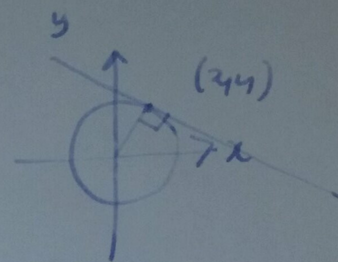
can't solve for y explicitly.

$(y(x))^4 + xy(x) = x^3 - x + 2 \leftarrow$ differentiate wrt x using chain rule on $y(x)$.

$$4(y(x))^3 \cdot y'(x) + y(x) + xy'(x) = 3x^2 - 1$$

$$y'(x)\left[4(y(x))^3 + x\right] = 3x^2 - 1 - y(x)$$

$$y'(x) = \frac{3x^2 - 1 - y}{4y^3 + x}$$



Example $x^2 + y^2 = 1$ $\quad 2x + 2yy' = 0 \quad y' = -x/y$

Application: derivatives of inverse functions.

examples $y = \ln(x)$

$$e^y = x$$

$$e^y \cdot y' = 1$$

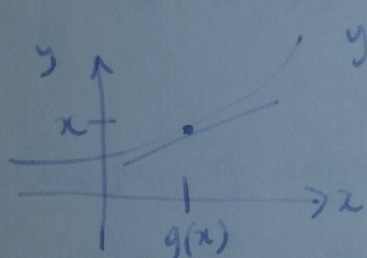
$$y' = e^{-y} = e^{-\ln(x)} = \frac{1}{x}$$

$$\boxed{\text{Thm } \frac{d}{dx}(\ln(x)) = \frac{1}{x}}$$

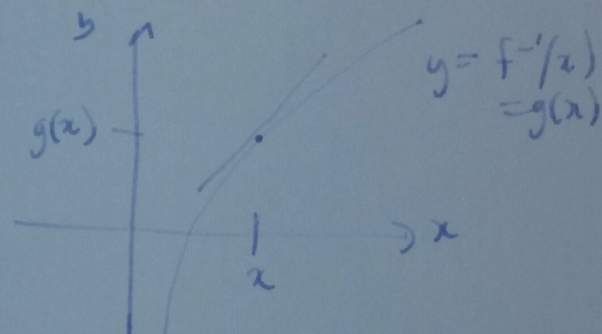
Thm suppose $f(x)$ differentiable, one-to-one, with inverse $f^{-1}(x) = g(x)$.

then $g'(x) = \dfrac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

recall:



$\xrightarrow{}$ reflect in $y = x$



$\frac{1}{\text{slope}}$ at $g(x)$ $\quad \longleftarrow \quad$ slope at x: $g'(x)$

i.e. $\dfrac{1}{f'(g(x))}$ $\square$.

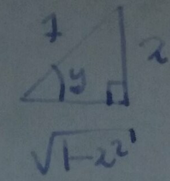
Application derivatives of inverse trig functions.

Thm $\dfrac{d}{dx}(\sin^{-1}(x)) = \dfrac{1}{\sqrt{1-x^2}}$, $\quad \dfrac{d}{dx}(\cos^{-1}(x)) = \dfrac{-1}{\sqrt{1-x^2}}$.

<u>Proof</u> (of $\sin^{-1}(x)$)

$$y = \sin^{-1}(x)$$

$$\sin(y) = x$$



$$\cos(y)\, y' = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad \square.$$

<u>Thm</u> $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$ $\qquad \frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{1+x^2}$

$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}}$, $\qquad \frac{d}{dx}\left(\csc^{-1}(x)\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

<u>Proof</u> (of $\tan^{-1}(x)$)

$$y = \tan^{-1}(x)$$

$$\tan(y) = x$$

$$\sec^2(y)\, y' = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \quad \square.$$

<u>Example</u> $\frac{d}{dx}\left(\tan^{-1}(e^{2x})\right) = \frac{1}{1+(e^{2x})^2} \cdot e^{2x} \cdot 2$

<u>§3.9 Derivatives of exponentials and logs</u>

recall: $f(x) = b^x$ then $f'(x) = m_b\, b^x$

$f(x) = e^x$ then $f'(x) = e^x$

<u>Thm</u> $f(x) = b^x$ then $f'(x) = \ln(b)\, b^x$

<u>Proof</u> $f(x) = b^x = e^{\ln(b)x}$ $\qquad f'(x) = e^{\ln(b)x} \cdot \ln(b) = b^x \cdot \ln(b)$ $\square$

<u>Example</u> $f(x) = 3^{4x} = (3^4)^x$, $f'(x) = \ln(3^4)(3^4)^x = 4\ln(3)\, 3^{4x}$.

<u>Thm</u> $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$.