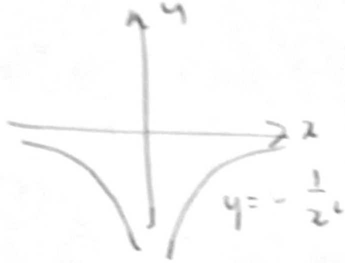
$$= \lim_{h\to 0} \frac{1}{h} \frac{-h}{(x+h)x} = \lim_{h\to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}.$$



Remarks

① functions: f: domain $\to$ range, e.g. $f: \mathbb{R} \to \mathbb{R}$

derivative : functions (differentiable) $\longrightarrow$ functions

$f(x) \longmapsto f'(x)$

warning : not all functions differentiable.

② "calculus" means rules for doing calculations, we won't have to explicitly compute limits all the time.

Example $f(x) = x^3$ $\quad f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2.$

Thm (powers of x) if $f(x) = x^n$, then $f'(x) = nx^{n-1}$ Fact works for all $n \in \mathbb{R}$!

Examples $\frac{d}{dx}(1) = \frac{d}{dx}(x^0) = 0 \cdot x^{-1} = 0.$

$\frac{d}{dx}(x^1) = \frac{d}{dx}(1x^0) = 1 \quad \frac{d}{dx}(x^2) = 2x \quad \frac{d}{dx}(x^3) = 3x^2.$

$\frac{d}{dx}(x^4) = 4x^3 \quad \frac{d}{dx}(x^{100}) = 100x^{99}, \quad \frac{d}{dx}(x^{-1}) = \frac{d}{dx}\left(\frac{1}{x}\right) = -x^{-2} = \frac{-1}{x^2}.$

$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}.$

Proof let $f(x) = x^n$, $\quad f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

binomial theorem : $(x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + h^n.$

$$(x+h)^n = x^n + nx^{n-1}h + \underbrace{\binom{n}{2}x^{n-2}h^2 + \cdots + h^n}_{\text{all have powers of } h \geq 2}$$

so $\displaystyle\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n - x^n}{h}$

$\displaystyle = \lim_{h\to 0} nx^{n-1} + h(\text{—}) = nx^{n-1}$. $\square$

<u>warning</u>: this rule works for polynomials only, <u>not</u> exponentials.

$f(x) = x^{100}$ polynomial $\qquad f(x) = 2^x$ <u>not</u> polynomial.

<u>Other useful rules</u>

<u>Thm</u> (Linearity) If f, g differentiable functions, then $f+g$ is differentiable with $(f+g)' = f' + g' \leftrightarrow \frac{d}{dx}(f+g) \leftrightarrow \frac{df}{dx} + \frac{dg}{dx}$

If k is constant: $(kf)' = kf' \leftrightarrow \frac{d}{dx}(kf) = k\frac{df}{dx}$.

<u>Proof</u> (follows from limit laws)

$\displaystyle (f+g)'(x) = \lim_{h\to 0}\frac{(f+g)(x+h) - (f+g)(x)}{h}$

$\displaystyle = \lim_{h\to 0}\frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$

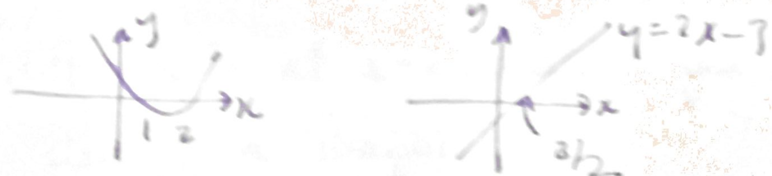
$\displaystyle = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)$.

$\displaystyle (kf)'(x) = \lim_{h\to 0}\frac{kf(x+h) - kf(x)}{h} = k\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = kf'(x)$.

<u>Example</u> $f(x) = x^2 - 3x + 2$ find $f'(x)$

$\displaystyle \frac{df}{dx} = \frac{d}{dx}(x^2 - 3x + 2) = \frac{d}{dx}(x^2) - 3\frac{d}{dx}(x) + \frac{d}{dx}(2) = 2x - 3 + 0$

<u>graphs</u> $f(x) = (x-2)(x-1)$



Derivative of e^x

consider $f(x) = b^x \quad b>0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h}-b^x}{h} = \lim_{h\to 0} b^x \frac{(b^h-1)}{h}.$$

$$= b^x \underbrace{\lim_{h\to 0} \frac{b^h-1}{h}}$$ doesn't depend on x! → assume this limit exists and call it m_b.

we have shown: for exponential functions the derivative is proportional to the value of the function, i.e. $f(x) = b^x$, then $f'(x) = m_b b^x$

in particular, at $x=0$ slope is $f'(0) = m_b$

recall: e is the special number s.t. the slope of e^x at $x=0$ is equal to 1

therefore if $\boxed{f(x) = e^x, \quad f'(x) = e^x}$ ⇔ $\boxed{\frac{d}{dx}(e^x) = e^x}$

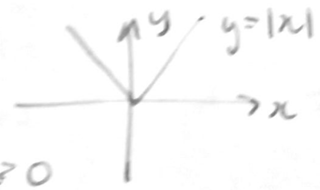
Example $\frac{d}{dx}\left(7e^x + 4x^2\right) = 7e^x + 8x.$

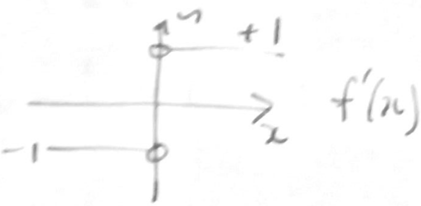
observation: this shows that e^x is not a polynomial.

$\frac{d}{dx}(p(x))$ ← degree goes down, eventually differentiates to 0.

Thm Differentiable ⇒ continuous. Warning: continuous ⇏ differentiable.

example $f(x) = |x|$ cts





but not differentiable at $x=0$

check $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|x+h|-|x|}{h}$ when $x=0$ $\frac{0}{0}$ undefined

$\lim_{h\to 0+} \frac{|x+h|-|x|}{h} = \lim_{h \to 0} \frac{|h|}{h} = 1 \qquad \lim_{h\to 0-} \frac{|x+h|-|x|}{h} = \frac{-h}{h} = -1.$

$1 \neq -1$ so $\lim_{h\to 0} \frac{|x+h|-|x|}{h}$ DNE.

local picture if $f(x)$ is differentiable at $x=c$, then if you look close enough, the graph looks close to a straight line.

Proof (differentiable ⇒ cts) $f(x)$ differentiable at $x=c$ means $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists. (want to show $\lim_{h\to 0} f(c+h) = f(c)$).

consider $f(c+h) - f(c) = h\left(\frac{f(c+h)-f(c)}{h}\right)$ so $\lim_{h\to 0} f(c+h) - f(c) =$

$\lim_{h\to 0} h \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = 0 \cdot f'(c) = 0$. □.

§3.3 Product rules and quotient rules

new functions from old: $f(x)g(x)$ product $\qquad \frac{f(x)}{g(x)}$ quotient

Thm (product rule) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

warning: $(fg)' \neq f'g'$!!

Examples ① $\frac{d}{dx}(x^2) = \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) = 1x + x \cdot 1 = 2x$.

② $\frac{d}{dx}\left(3x^2(x^2+1)\right) = (3x^2)'(x^2+1) + 3x^2(x^2+1)' = 6x(x^2+1) + 3x^2 \cdot 2x$.

③ $\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2) \cdot e^x + x^2\frac{d}{dx}(e^x) = 2xe^x + x^2e^x$

Proof (of product rule) (assume f, g differentiable at x)

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}.$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \frac{f(x+h)-f(x)}{h}g(x)$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} g(x) = f(x)g'(x) + f'(x)g(x)$$

□.