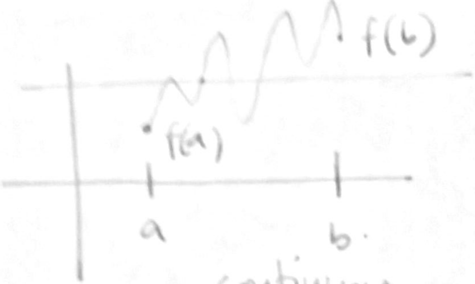
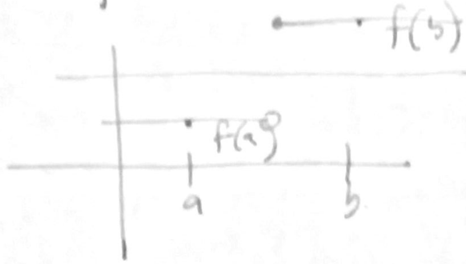
§2.8 Intermediate value theorem (IVT)

"continuous functions can't skip values"



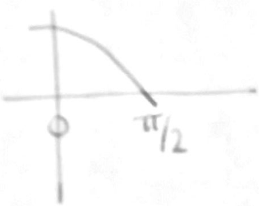


continuous

Thm (Intermediate value Theorem (IVT))

If $f(x)$ is a continuous function on a closed interval $[a,b]$ and $f(a) \neq f(b)$ then for any number M between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ s.t. $f(c) = M$. □.

Example show $\cos(x) = \frac{1}{4}$ has at least one solution



consider $[0, \frac{\pi}{2}]$ $\quad \cos(0) = 1$, $\cos(\frac{\pi}{2}) = 0$ $\quad 0 \le \frac{1}{4} \le 1$

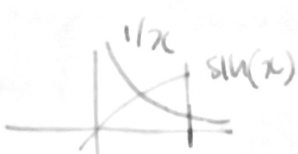
$\Rightarrow \exists\, c \in [0, \frac{\pi}{2}]$ s.t. $\cos(c) = \frac{1}{4}$.

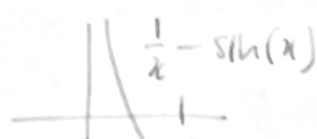
special case: finding zeros.

Corollary if $f(x)$ is continuous on $[a,b]$ and $f(a), f(b)$ have different signs, then there is at least one $c \in [a,b]$ s.t. $f(c) = 0$.

Bisection method: find a solution to $\sin(x) = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

consider $f(x) = \frac{1}{x} - \sin(x)$





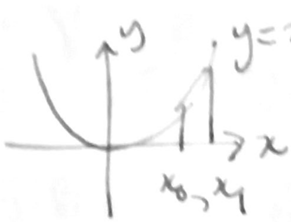
$f(0)$ undefined but $\to \infty$ as $x \to 0$

$f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 \simeq -0.36 < 0$ $\quad$ check midpoint: $\frac{\pi}{4}$ $\quad f(\frac{\pi}{4}) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \simeq 0.566 > 0$

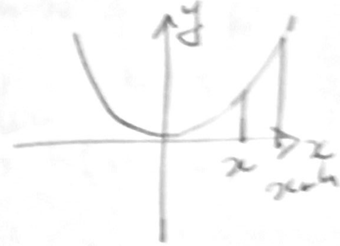
now continue with $[\frac{\pi}{4}, \frac{\pi}{2}]$ etc...

§3.1 Defn of the derivative

Recall: we can compute the average rate of change of a function over an interval



over $[x_0, x_1]$: $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$



Q: how do we compute slope of the tangent line?

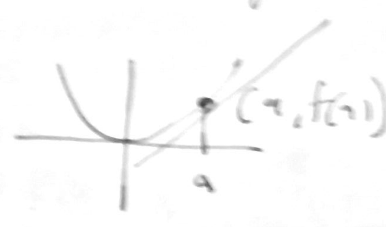
idea: look at average rate of change over small interval $[x, x+h]$ and take limit as $h \to 0$.

Defn the slope of the tangent line at $x=a$ is $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

Notation: also called the deriviative, written $f'(a)$ (Newton) or $\dfrac{df}{dx}(a)$ (Leibnitz)

note $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ same as $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$

Defn the tangent line to $f(x)$ at the point $(a, f(a))$ is the straight line through $(a, f(a))$ with slope $f'(a)$.



equation of line: $y-y_0 = m(x-x_0)$

i.e. $y - f(a) = f'(a)(x-a)$ or $y = f(a) + f'(a)(x-a)$

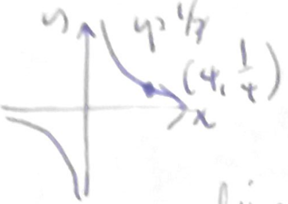
Example find tangent line to $y=x^2$ at $x=1$

$(x, f(x))$ is $(1,1)$ slope $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0} \dfrac{(1+h)^2-1^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{1+2h+h^2-1}{h} = \lim\limits_{h\to 0} \dfrac{2h+h^2}{h} = \lim\limits_{h\to 0} 2+h = 2$

so equation of tangent line is $y-1 = 2(x-1)$ or $y = 2x-1$

<u>Example</u> find slope of tangent line to $f(x) = \frac{1}{x}$ at $x = 4$.



slope $f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{4+h} - \frac{1}{4}\right) = \lim_{h\to 0} \frac{1}{h} \frac{4-(4+h)}{(4+h)4} = \lim_{h\to 0} \frac{-h}{h(16+4h)} =$

$= \lim_{h\to 0} \frac{-1}{16+4h} = -\frac{1}{16}$ tangent line: $y - \frac{1}{4} = -\frac{1}{16}(x-4)$

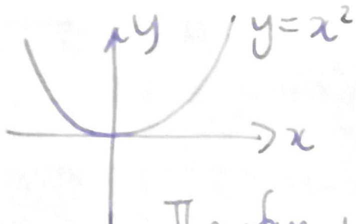
<u>Example</u> straight line $y = mx+b$. find slope at $x = a$.

$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{m(a+h)+b-(ma+b)}{h} = \lim_{h\to 0} \frac{\cancel{ma} + mh + \cancel{b} - \cancel{b} - \cancel{ma}}{h}$

$= \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m.$

<u>observation</u> if $f(x) = b$, constant, then $f'(x) = 0$ for all x.

<u>§3.2 Derivative as a function</u>



$f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$

at each input x, there is a tangent line, which has a slope, which is a number.

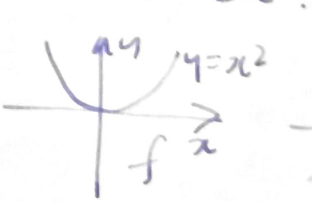
Therefore we can define a function $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto f'(x)$ slope of tangent line at x.

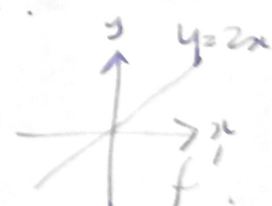
<u>notation</u>: we call this function $f'(x)$, or the "derivative of f"

<u>Example</u> if $f(x) = x^2$, then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$

$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} 2x + h = 2x.$

<u>summary</u> if $f(x) = x^2$, then $f'(x) = 2x$





<u>example</u> $f(x) = \frac{1}{x}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{x - (x+h)}{(x+h)(x)}$