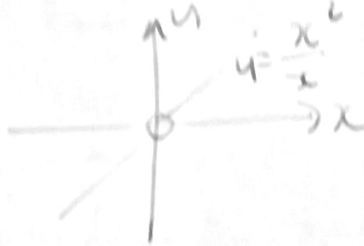
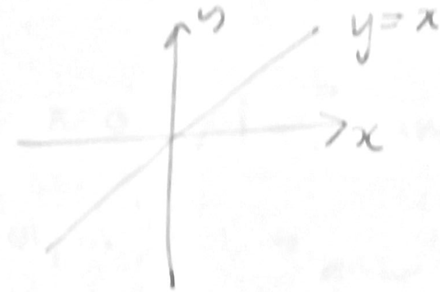
# §2.5 Evaluating limits algebraically

Example. $\frac{x^2}{x}$ undefined at $x=0$. $\frac{0}{0}$ indeterminate form.

but $\lim\limits_{x\to 0}\frac{x^2}{x}$ does not depend on value at $x=0$.

$\frac{x^2}{x}=x$ for $x\neq 0$




so $\lim\limits_{x\to 0}\frac{x^2}{x}=\lim\limits_{x\to 0}x = 0$

Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty\cdot 0$, $\infty-\infty$, $0^0$

note: $\frac{1}{0}$ not indeterminate, gives limit $\pm\infty$ or DNE.

Examples ① $\lim\limits_{x\to 3}\frac{x^2-4x+3}{x^2+x-12}$ $\quad x=3$: $\frac{9-12+3}{9+3-12}=\frac{0}{0}$

factor: $\frac{(x-3)(x+1)}{(x-3)(x+4)}=\frac{x-1}{x+4}$ $(x\neq 3)$

so $\lim\limits_{x\to 3}\frac{x^2-4x+3}{x^2+x-12}=\lim\limits_{x\to 3}\frac{x-1}{x+4}=\frac{2}{7}$

② $\lim\limits_{x\to 4}\frac{\sqrt{x}-2}{x-4}$: $\quad x=4$ $\frac{2-2}{4-4}=\frac{0}{0}$

$\frac{\sqrt{x}-2}{x-4}=\frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)}=\frac{1}{\sqrt{x}+2}$ $(x\neq 4)$ so $\lim\limits_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim\limits_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}$.

③ $\lim\limits_{x\to \frac{\pi}{2}}\frac{\tan x}{\sec x}$ $\quad \frac{\tan x}{\sec x}=\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}=\sin x$ $\quad \lim\limits_{x\to\frac{\pi}{2}}\sin x = 1$.

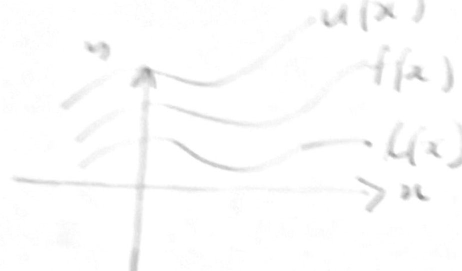
$$\lim_{x\to 1} \frac{1}{x-1} - \frac{2}{x^2-1} \qquad \infty - \infty \qquad \frac{1}{x-1} - \frac{2}{(x-1)(x+1)} = \frac{x+1-2}{(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}.$$

## §2.6 Trigonometric limits

Q: what is $\lim_{x\to 0} \frac{\sin x}{x}$ ? substitute $x=0$ get $\frac{0}{0}$ undefined.

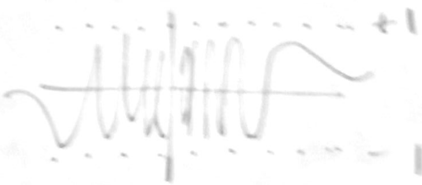
A: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (can try $x = 0.1, 0.01, \ldots$)

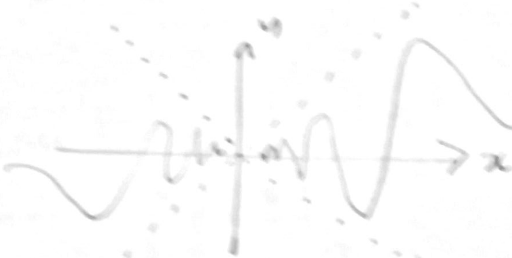



Squeeze theorem: suppose $l(x) \le f(x) \le u(x)$

Thm suppose $l(x) \le f(x) \le u(x)$ and $\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$
then $\lim_{x\to c} f(x) = L$.

Example $\lim_{x\to 0} x \sin(\frac{1}{x})$

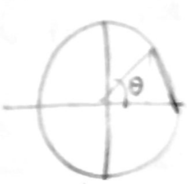


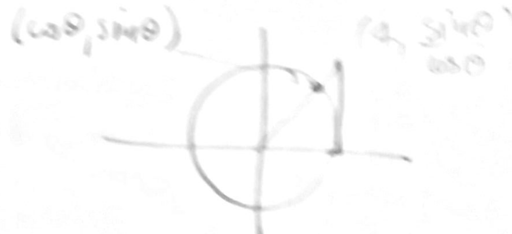




$-|x| \le x \sin(\frac{1}{x}) \le |x| \qquad \lim_{x\to 0} |x| = 0 \qquad \lim_{x\to 0} -|x| = 0 \Rightarrow \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$

Thm $\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Proof (of $\lim_{x\to 0} \frac{\sin x}{x} = 1$) assume $0 < \theta < \frac{\pi}{2}$ consider the following areas:




area of triangle $\le$ area of sector $\le$ area of triangle

$\frac{1}{2}bh$

$\frac{1}{2} \cdot 1 \cdot \sin\theta \le \pi r^2 \frac{\theta}{2\pi} \le \frac{1}{2} \frac{\sin\theta}{\cos\theta}$

$$\underbrace{\tfrac{1}{2}\sin\theta \le \tfrac{1}{2}\theta}_{\frac{\sin\theta}{\theta} \le 1} \le \tfrac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\theta} \le 1 \qquad\qquad \cos\theta \le \frac{\sin\theta}{\theta}$$

so $\cos\theta \le \dfrac{\sin\theta}{\theta} \le 1$ $\qquad \lim\limits_{\theta\to 0} \cos\theta = 1 \qquad \lim\limits_{\theta \to 0} 1 = 1$

squeeze theorem $\Rightarrow \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ $\square$.

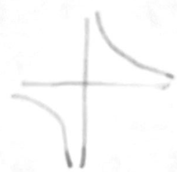
Examples ① $\lim\limits_{x\to 0} \dfrac{\sin 3x}{2x}$ $\qquad$ know: $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$.

write $3x = \theta$ $\qquad \lim\limits_{x\to 0} \dfrac{\sin\theta}{2(\theta/3)} = \lim\limits_{x\to 0} \dfrac{3}{2}\dfrac{\sin\theta}{\theta} = \dfrac{3}{2}\lim\limits_{x\to 0}\dfrac{\sin\theta}{\theta} = \dfrac{3}{2}$.

② $$\lim_{t\to 0}\frac{1-\cos t}{\sin t} = \lim_{t\to 0}\frac{1-\cos t}{t}\cdot\frac{t}{\sin t} = \underbrace{\lim_{t\to 0}\frac{1-\cos t}{t}}_{0}\cdot\underbrace{\lim_{t\to 0}\frac{t}{\sin t}}_{1} = 0$$

## §2.7 Limits at infinity

key observations: $\lim\limits_{x\to\infty}\dfrac{1}{x} = 0$



$\lim\limits_{x\to\infty} x = +\infty$



Examples

① $$\lim_{x\to\infty}\frac{3x}{2x-1} = \lim_{x\to\infty}\frac{3}{2-1/x} = \frac{3}{2}$$

② $$\lim_{x\to\infty}\frac{x^2+x}{x-3} = \lim_{x\to\infty}\frac{x+1}{1-3/x} = \lim_{x\to\infty} x+1 = \infty.$$

③ $$\lim_{x\to\infty}\frac{1}{x} - \frac{2}{3x+1} = \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{2}{3x+1} = 0 - 0 = 0.$$

④ $$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{4x+1} = \lim_{x\to\infty}\frac{\sqrt{2+1/x^2}}{4+1/x} = \frac{\sqrt{2}}{4}$$