<u>Thm</u> assume that $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist and are finite. Then:

1) sums: $\lim_{x\to c} (f(x)+g(x)) = \left(\lim_{x\to c} f(x)\right) + \left(\lim_{x\to 0} g(x)\right)$

2) constant multiple: $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$ $\quad$ k constant (doesn't depend on x)

3) products: $\lim_{x\to c} (f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

4) quotients: $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ as long as $\lim_{x\to c} g(x) \neq 0$.

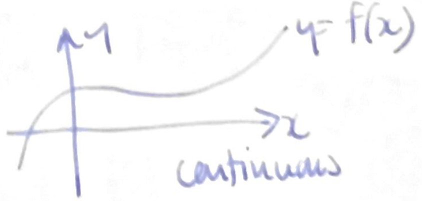
<u>Warning</u>: these rules do not work if either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ DNE.

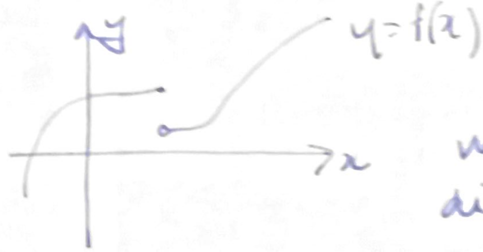
<u>Examples</u> $\lim_{x\to 3} x^2 = \lim_{x\to 3} x \cdot \lim_{x\to 3} x = 3\cdot 3 = 9$.

$\lim_{t\to 2} \frac{t+5}{3t} = \frac{\lim_{t\to 2} t+5}{\lim_{t\to 2} 3t} = \frac{7}{6}$

<u>§2.4 Limits and continuity</u>

<u>Example</u>



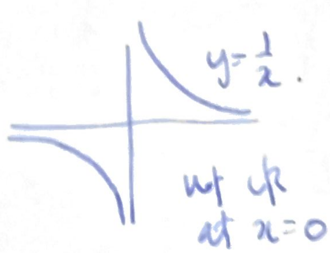


not continuous / discontinuous

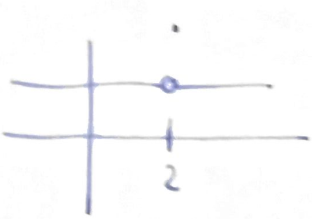
<u>Defn</u> we say $f(x)$ is <u>continuous</u> at $x=c$ if $\lim_{x\to c} f(x) = f(c)$

if the limit DNE / is infinite / is not equal to $f(c)$, then $f(x)$ is not continuous at c.

<u>Examples</u>



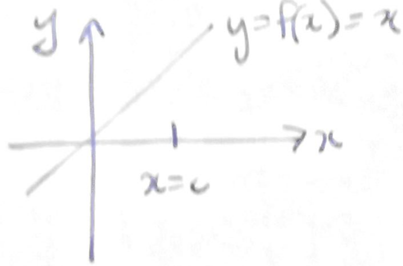
not cts at $x=0$.



$f(x) = \begin{cases} 1 & x \neq 2 \\ 2 & x = 2 \end{cases}$

not continuous at $x=2$

Example show $f(x) = x$ is continuous



$f(c) = c$

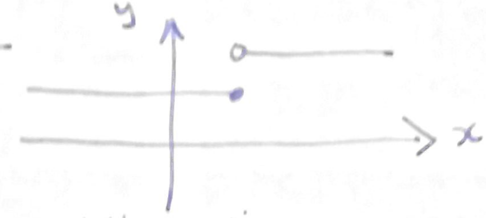
want to show $\lim_{x \to c} f(x) = f(c)$

follows from limit law $\lim_{x \to c} x = c$

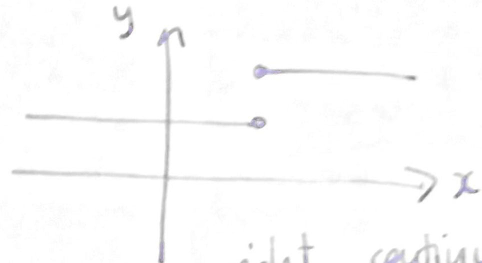
Def^n $f(x)$ is left continuous at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$

right continuous " if $\lim_{x \to c^+} f(x) = f(c)$

Examples



left continuous



right continuous

If at least one of the left or right limits is $\pm\infty$ we say $f(x)$ has an infinite discontinuity at $x = c$.

Building continuous functions

Thm 0: $f(x) = k$, $f(x) = x$ are continuous

Thm 1: suppose that $f(x)$, $g(x)$ are both cts at $x = c$. Then the following functions are cts at $x = c$:

1) $f(x) + g(x)$
2) $k f(x)$ k constant
3) $f(x) g(x)$
4) $\dfrac{f(x)}{g(x)}$ if $g(c) \neq 0$

Proof: these follow directly from the limit laws.

check: 1) $f(x)$ cts means $\lim_{x \to c} f(x) = f(c)$

$g(x)$ cts means $\lim_{x \to c} g(x) = g(c)$

so $\lim_{x \to c} \left(f(x) + g(x) \right) = \lim_{x \to c} f(x) + \lim_{x \to x} g(x) = f(c) + g(c)$
as required $\square$.

Thm 2 Polynomials are continuous $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

Rational functions $\frac{p(x)}{q(x)}$ are continuous, except where $q(x) = 0$.

Proof $f(x) = x$ is continuous, so $f(x) \cdot f(x) = x \cdot x = x^2$ is cts.
similarly x^n is continuous, so $p(x) = a_n x^n + \dots + a_0$ is cts
(multiply by constants, add)
so $\frac{p(x)}{q(x)}$ is cts (quotients) where $q(x) \neq 0$. $\square$.

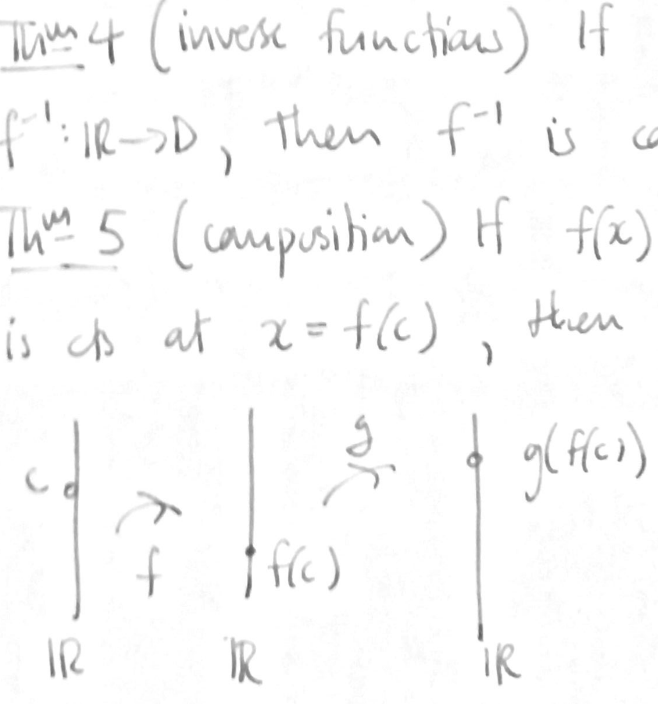
useful facts

Thm 3
- $\sin(x)$, $\cos(x)$ are cts.
- b^x is cts.
- $\log_b(x)$ is cts
- $x^{1/n}$ is cts.

} combinations of these and polynomials are sometimes called elementary functions.

Thm 4 (inverse functions) If $f : D \to \mathbb{R}$ ($D \subseteq \mathbb{R}$) is cts, with inverse $f^{-1} : \mathbb{R} \to D$, then f^{-1} is continuous.

Thm 5 (composition) If $f(x)$ is continuous at $x = c$, and $g(x)$ is cts at $x = f(c)$, then $g(f(x))$ is cts at $x = c$



Example $f(x) = \frac{2^x + \sin(x)}{\sqrt{x^2 + x + 1}}$, cts at $x = 1$

Q: where is $\frac{x^2}{\sin(x)}$ cts?