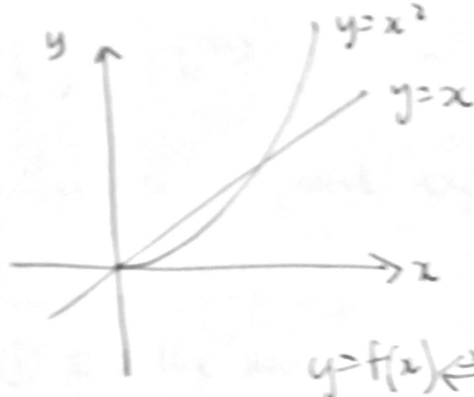
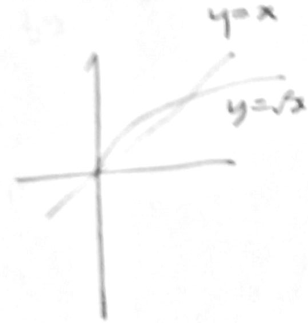
How to draw the graph of the inverse



A: reflect in $y=x$

reason: graph of f: $(x, f(x))$

f^{-1}: $(x, f^{-1}(x))$



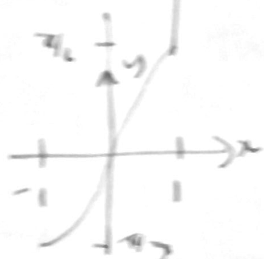
$y=f(x) \Leftrightarrow f^{-1}(y)=x \qquad (x, f(x)) \leftrightarrow (f^{-1}(y), y)$.

swap and relabel.

Inverse trig functions

$y=\sin(x)$

$y=\sin^{-1}(x)$



problem: not one-to one.

fix: restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

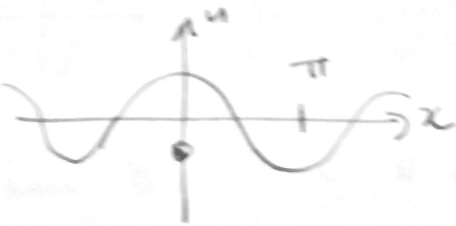
$\sin(x)$: $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$

$\sin^{-1}(x)$: $[-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\parallel$

$\operatorname{arcsin}(x) = \operatorname{asin}(x)$.

similarly $y=\cos(x)$



restrict domain to $[0,\pi]$

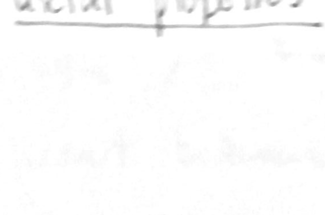
$\cos(x)$: $[0,\pi] \to [-1,1]$

$\cos^{-1}(x)$: $[-1,1] \to [0,\pi]$

§1.6 Exponential and Logarithm functions

Example $x \mapsto 2^x$

x	-2	-1	0	1	2	3
$f(x)$	$1/4$	$1/2$	1	2	4	8



can use any positive number instead of 2 $\quad f(x)=b^x$ $(b>0)$

useful properties:
- positive $b^x > 0$ for all x
- b^x increasing if $b>1$
- b^x decreasing if $0<b<1$
- b^x grows faster than any polynomial x^n

exponent rules : $b^0 = 1$ $\quad b^x b^y = b^{x+y}$ $\quad b^{-x} = \frac{1}{b^x}$ $\quad \frac{b^x}{b^y} = b^{x-y}$

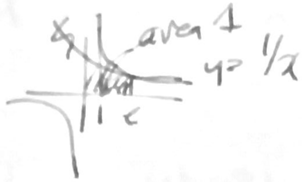
$(b^x)^y = b^{xy}$ $\quad b^{1/n} = \sqrt[n]{b}$

- there is a special exponential function e^x , $e = 2.71828\ldots$

key properties

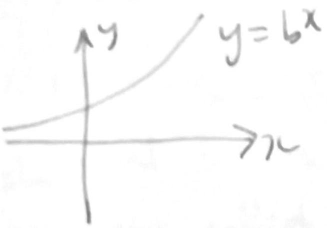
① e is the unique number s.t. e^x has slope 1 at $x = 0$

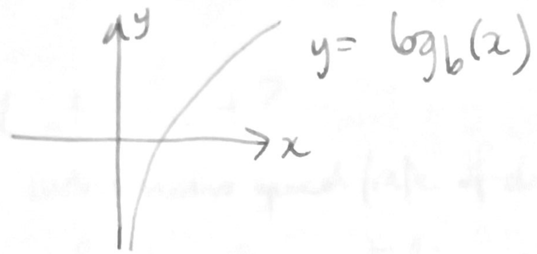
② e is the unique number s.t. the area under the curve $y = 1/x$ between 1 and e has area 1



logarithms

the logarithm is the inverse function for the exponential function.





the special logarithm with base $b = e$ is called the natural logarithm $\ln(x)$

- inverse function properties: $f^{-1}(f(x)) = x = f(f^{-1}(x))$

so $\quad b^{\log_b(x)} = x = \log_b(b^x)$

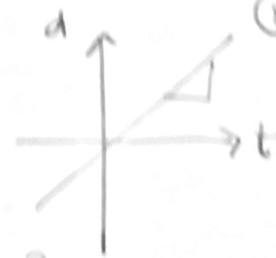
- logarithm rules : $\log_b(1) = 0$, $\log_b(b) = 1$

$\log_b(st) = \log_b(s) + \log_b(t)$ $\quad \log_b(1/t) = -\log_b(t)$ $\quad \log_b(s/t) = \log_b(s) - \log_b(t)$

$\log_b(s^t) = t\log_b(s)$

convert between different bases: $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$ for any a in particular $= \frac{\ln(x)}{\ln(b)}$

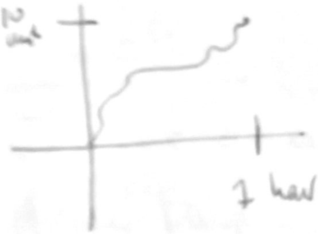
§2.1 Limits, rates of change, tangent lines



motivation: velocity, example driving at constant speed

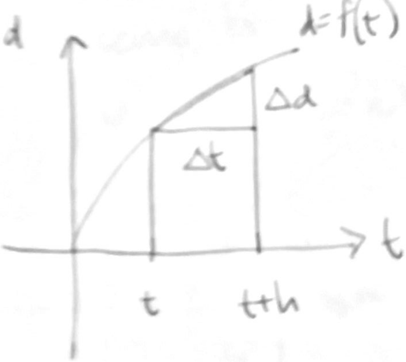
velocity $= \dfrac{\text{distance}}{\text{time}}$ = slope of line

problem: what happens if you don't travel at constant speed?



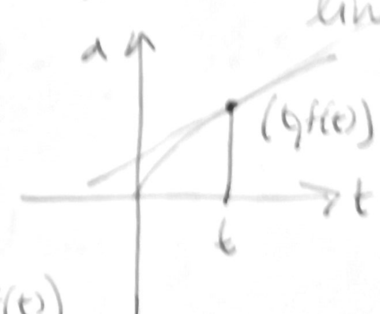
average speed $= \dfrac{\text{distance travelled}}{\text{time taken}}$

we can look at average speed over any time interval, including very short ones.



average speed on interval $[t, t+h]$

is $\dfrac{\Delta d}{\Delta t} = \dfrac{f(t+h) - f(t)}{(t+h) - t} = \dfrac{f(t+h) - f(t)}{h}$.



Q: what is the speed at time t?
(sometimes called the instantaneous speed (rate of change))

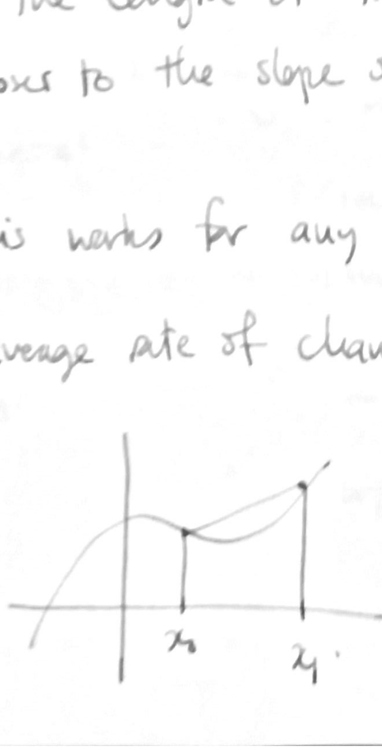
A: speed is the slope of the tangent line at $(t, f(t))$

idea/hope: as the length of the interval $[t, t+h]$ gets small, the average speed gets closer to the slope of the tangent line. This works for "nice" functions.

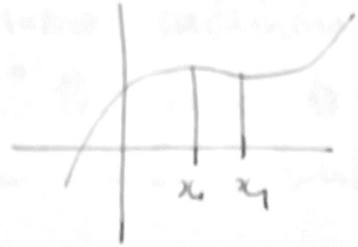
observation: this works for any function $y = f(x)$, not just speed.

summary average rate of change over an interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



§2.2 Limits



aim: find slope of tangent lines.

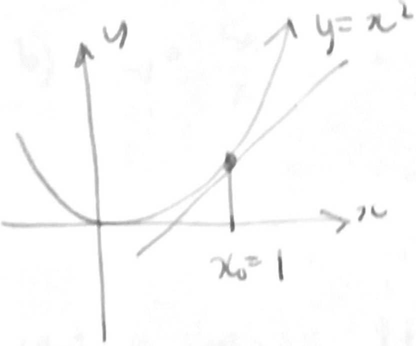
know: average rate of change $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$

Q: why can't we just set $x_0 = x_1$? A: doesn't work, get $\dfrac{f(x_0)-f(x_0)}{x_0-x_0} = \dfrac{0}{0}$ undefined!

Observations

① if we draw careful pictures, the average slope gets closer to the slope of the tangent line as the length of the interval gets small.

② seems to work for sample calculation too:



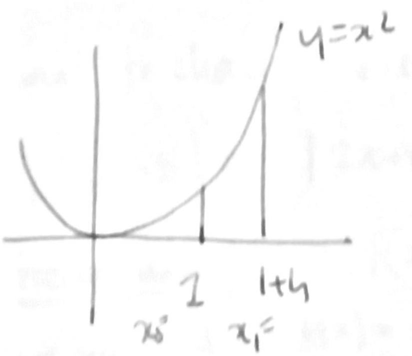
$x_1 = 2$: $\dfrac{f(2)-f(1)}{2-1} = \dfrac{4-1}{1} = 3$

$x_1 = 1.5$: $\dfrac{f(3/2)-f(1)}{3/2-1} = \dfrac{9/4-1}{1/2} = \dfrac{5}{2} = 2.5$

$x_1 = 1.1$: $\dfrac{1.21-1}{0.1} = 2.1$

$x_1 = 1.01$: $\dfrac{1.0201-1}{0.01} = 2.01$

③ seems to work algebraically:

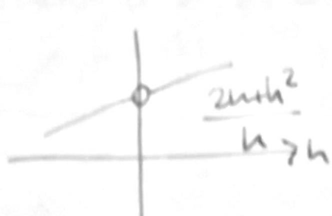


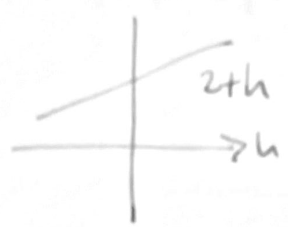
average rate of change from $\underset{x_0}{1}$ to $\underset{x_1}{1+h}$

$$= \frac{f(1+h)-f(1)}{1+h-1} = \frac{(1+h)^2-1^2}{h} = \frac{1+2h+h^2-1}{h}$$

$$= \frac{2h+h^2}{h} = 2+h$$

$(h \neq 0!)$



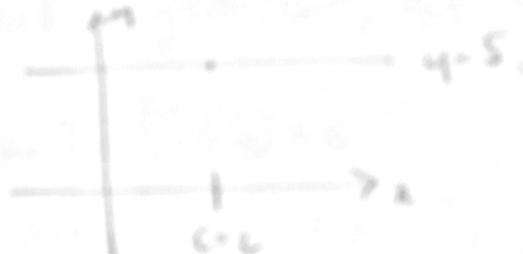


Def^n Let f be a function defined on an interval containing c, but not necessarily defined at c. We say "the limit of $f(x)$ as x approaches c is equal to L" if $|f(x) - L|$ becomes arbitrarily small as x gets close to c.

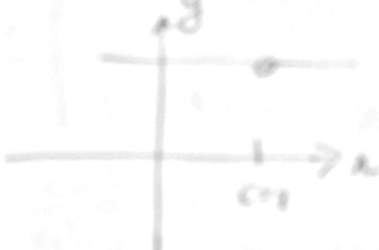
Notation: $\lim_{x \to c} f(x) = L$ or $f(x) \to L$ as $x \to c$.

We also say "$f(x)$ converges to L as x tends to c"

Examples a) $f(x) = 5$, $c = 2$



b) $f(x) = \frac{5x}{x}$, $c = 2$



want to show: $|f(x) - 5|$ close to 0 if x close to 2

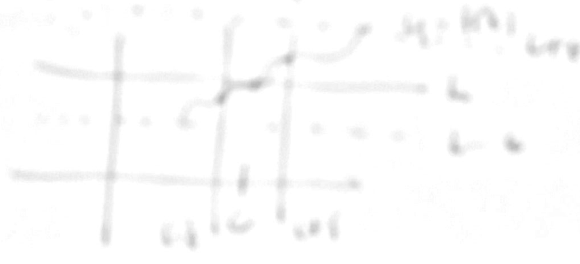
$|f(x) - 5| = |5 - 5| = 0$ for all $x \neq 2$, so this holds.

c) $\lim_{x \to 2} 2x + 1 = 5$

want to show $|f(x) - 5|$ close to 0 when x close to 2.

$|f(x) - 5| = |2x + 1 - 5| = |2x - 4| = 2|x - 2|$. x close to $2 \Rightarrow |x - 2|$ close to 0.

Precise def^n: Let $f(x)$ be defined on an interval containing c, but not nec. at c. We say $\lim_{x \to c} f(x) = L$, if for all $\epsilon > 0$, there is a $\delta > 0$ s.t. if $|x - c| < \delta$ then $|f(x) - L| < \epsilon$



<u>useful facts</u>

<u>Thm:</u> for any constants k, c

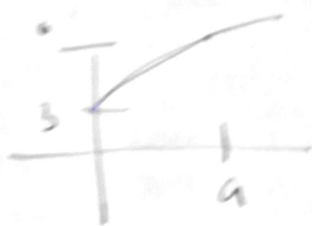
$$\lim_{x \to c} k = k$$

$$\lim_{x \to c} x = c$$

<u>investigating limits:</u> try:
- drawing a picture
- calculating close values
- algebra

<u>Example</u> $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$ problem: can't plug in $x=9$, get $\frac{0}{0}$.

- draw picture



looks like $f(6) = 6$.

- calculate:

x	$\frac{x-9}{\sqrt{x}-3}$
8.9	5.983
8.99	5.998
9.01	6.002
9.1	6.016

- algebra: difference of two squares:

$$(a^2 - b^2) = (a+b)(a-b).$$

$$x - 9 = (\sqrt{x}-3)(\sqrt{x}+3).$$

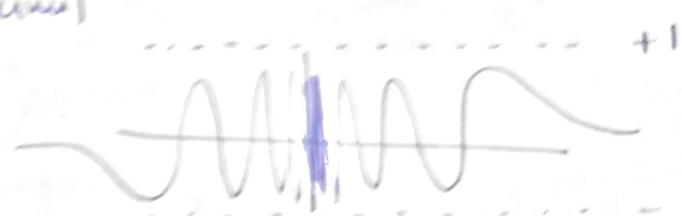
$$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} = \sqrt{x}+3 \quad (x \neq 9!)$$

$$\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \to 9} \sqrt{x}+3 = 6.$$

<u>Bad example</u>: no limit

$f(x) = \sin\left(\frac{1}{x}\right)$

no limit at $x = 0$



<u>idea</u>: $f\left(\frac{1}{2\pi n}\right) = \sin(2\pi n) = 0$

$f\left(\frac{1}{2\pi n + \frac{\pi}{2}}\right) = \sin\left(2\pi n + \frac{\pi}{2}\right) = 1$

one sided limits example $f(x) = \frac{x}{|x|}$

$f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$

sometimes useful to distinguish left limit / right limit / two sided limit

notation $\lim_{x \to 0+} f(x)$ means right limit (only consider $x > 0$)

$\lim_{x \to 0-} f(x)$ means left limit (only consider $x < 0$)

note the two sided limit exists, the right limit must exist and equal the left limit : $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x)$.

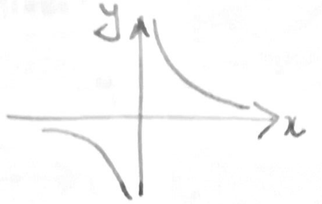
Example $f(x) = \frac{x}{|x|}$ $\left.\begin{array}{l} \lim_{x \to 0+} f(x) = 1 \\ \lim_{x \to 0-} f(x) = -1 \end{array}\right\}$ $1 \neq -1$ so $\lim_{x \to 0} \frac{x}{|x|}$ DNE.

Infinite limits

we say $\lim_{x \to c} f(x) = +\infty$ if $f(x)$ becomes arbitrarily large and positive as $x \to c$

$\lim_{x \to c} f(x) = -\infty$ if $f(x)$ " " " and negative as $x \to c$

Example $f(x) = \frac{1}{x}$

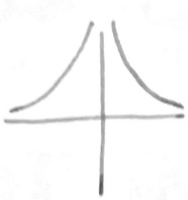


$\lim_{x \to 0+} \frac{1}{x} = +\infty$

$\lim_{x \to 0-} \frac{1}{x} = -\infty$

$\lim_{x \to 0} \frac{1}{x}$ DNE.

Example $f(x) = \frac{1}{x^2}$



$\lim_{x \to 0} \frac{1}{x^2} = +\infty$

§2.3 Basic limit laws

Example $\lim_{x \to 0} 2x+2 = \lim_{x \to 0} 2x + \lim_{x \to 0} 2 = 0 + 2 = 2$