MTH 231 Calculus I

Joseph Maher joseph.maher@csi.cuny.edu

web page http://www.math.csi.cuny.edu/~maher

office hours M 5:30–6:20 W 4:40–6:20

- students w/ disabilities.
- tutoring.

Text: Calculus, Rogawski, et al 4th edition.

HW: webworks : pdfs. / exams.

§1.2 Linear and quadratic functions

recall: input set (domain) $\xmapsto{\text{function}}$ output set (range)

examples: $f: \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ = real numbers)

$x \mapsto x^2$ or $f(x) = x^2$ (description of function)

e.g: $0 \mapsto 0$, $1 \mapsto 1$, $-1 \mapsto 1$, $2 \mapsto 4$, $3 \mapsto 9$ etc.

notation: $\underset{\text{name}}{f}: \underset{\text{domain}}{\mathbb{R}} \to \underset{\text{range}}{\mathbb{R}}$

Examples $+ : \mathbb{R}^2 \to \mathbb{R}$.
$(a,b) \mapsto a+b$

evaluation at 0 : $\underset{f:\mathbb{R}\to\mathbb{R}}{\{\text{functions}\}} \to \mathbb{R}$

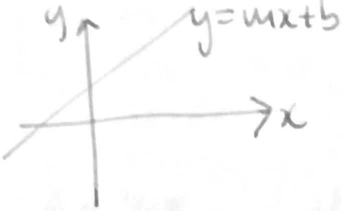
$f \mapsto f(0)$.

key property : every input gives ~~a~~ exactly one ~~unique~~ output $f: \mathbb{R} \to \mathbb{R}$, $1 \mapsto \{\pm 1\}$ X

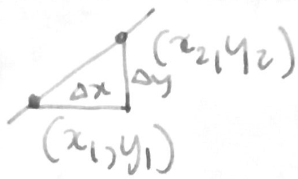
A linear function $f: \mathbb{R} \to \mathbb{R}$ has the form $f(x) = mx + b$

(m, b "constants", i.e. don't depend on x)

The graph of a linear function is a straight line.

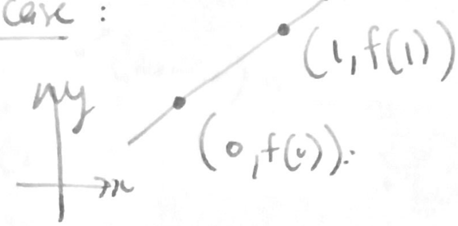


$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$$



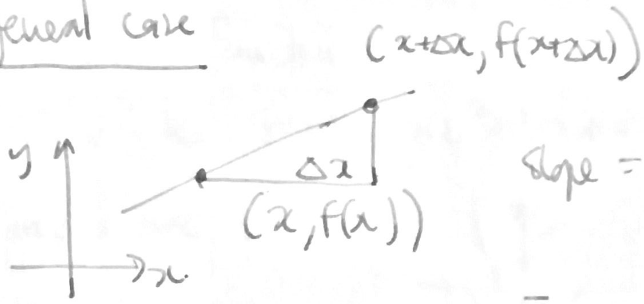
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

special case:



$$\frac{\Delta y}{\Delta x} = \frac{f(1) - f(0)}{1 - 0} = \frac{m + b - b}{1} = m.$$

general case

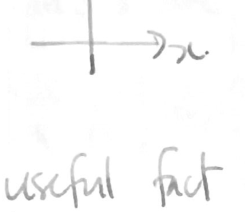


$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{x + \Delta x - x}$$

$$= \frac{m(x+\Delta x) + b - (mx + b)}{\Delta x} = \frac{m\Delta x}{\Delta x} = m.$$

useful fact: a straight line has constant slope everywhere.

observations:
- $|m|$ large steep slope \ / (−ve, +ve)
- $m = 0$, horizontal line
- $m > 0$ increasing (from left to right)
- $m < 0$ decreasing
- vertical lines not graphs of functions.



← equation of vertical line is $x = c$

equation vs function

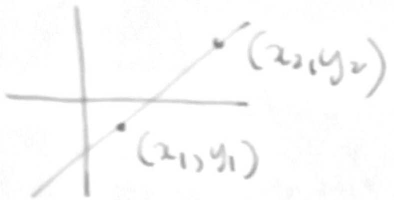
relation between variables, e.g. $x = c$, $x^2 + y^2 = 1$

a map from one set to another, eg. $f: \mathbb{R} \to \mathbb{R}$.

the graph of $f : \mathbb{R} \to \mathbb{R}$ is $y = f(x)$ (an equation!) but not every equation comes from a function, e.g. $x^2+y^2=1$ 

<u>note</u> to deal with <u>any</u> straight line, use the <u>general linear equation</u> $ax+by=c$ (at least one of a,b non-zero) e.g. $x=c$ set $b=0, a=1$.

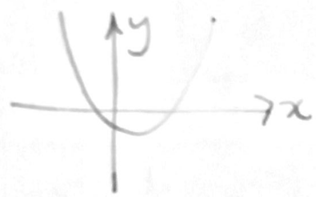
<u>useful technique</u> find equation of line through two points.

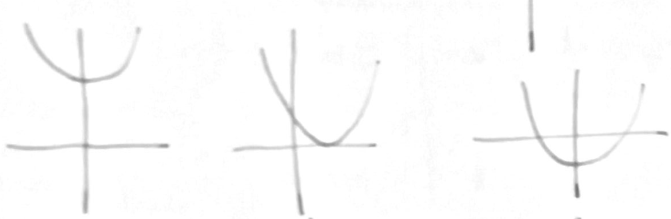


- find slope $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = m$
- line $y - y_1 = m(x - x_1)$ ← point slope formula for line.

<u>Quadratic functions</u>

given by $f(x) = ax^2 + bx + c$ (a,b,c constants, do not depend on x)

graphs are parabolas  at most two distinct real solutions to $f(x)=0$ given by $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

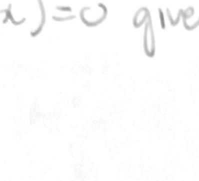


$b^2-4ac<0$ $\quad$ $b^2-4ac=0$ $\quad$ $b^2-4ac>0$.

<u>useful techniques</u>

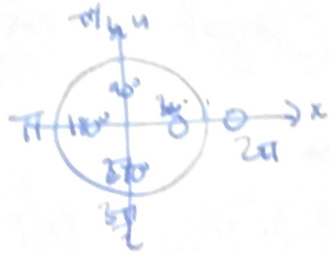
- factorization: $ax^2+bx+c = a(x-r_1)(x-r_2)$, then r_1, r_2 are roots / solutions to $f(x)=0$.
- complete the square: any quadratic function can be written as $a(x+b)^2+c$

example: x^2+2x+3

$(x+1)^2+1$ ← no roots! 

$x^2+2x+2+1$

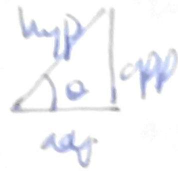
§1.4 Trig functions



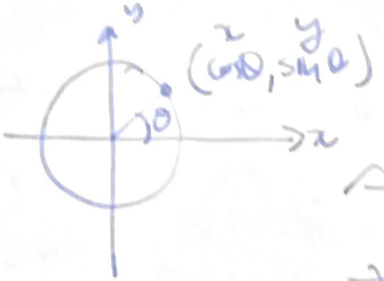
- angles vs radians : radians win.

angle in radians = distance travelled around unit circle.

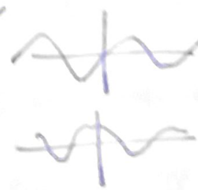
- right angled triangles



$\sin\theta = \dfrac{opp}{hyp}$ $\cos\theta = \dfrac{adj}{hyp}$ } can extend these function to be defined for all $\theta \in \mathbb{R}$.

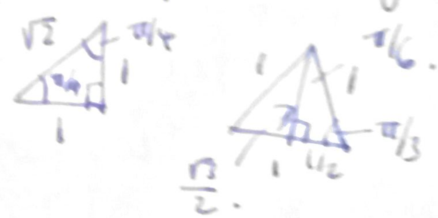


useful facts : $\sin(-\theta) = -\sin(\theta)$ (odd function)
$\cos(-\theta) = \cos(\theta)$ (even function)



sinθ, cosθ } graphs periodic w/ period 2π.

from special triangles



special values

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin θ	0	$\sqrt{1}/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{4}/2$
cos θ	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0

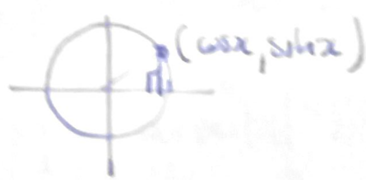
other trig functions

$\tan(x) = \dfrac{opp}{adj} = \dfrac{\sin x}{\cos x}$ $\quad \operatorname{cosec}(x) = \dfrac{1}{\sin x}$ $\quad \sec x = \dfrac{1}{\cos x}$

$\cot(x) = \dfrac{1}{\tan x}$.

Trig identities

Pythagorean identity $\cos^2 x + \sin^2 x = 1$



$\dfrac{\cos^2 x + \sin^2 x}{\sin^2 x} = \dfrac{1}{\sin^2 x} \iff \cot^2 x + 1 = \operatorname{cosec}^2 x$

$\dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} \iff 1 + \tan^2 x = \sec^2 x$.

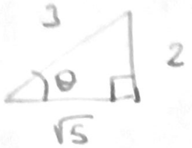
Addition.

$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$.

Double angle
$\sin 2\theta = 2 \sin\theta \cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

special case (shift) $\sin(x+\frac{\pi}{2}) = \cos x$

Example suppose $\sin\theta = \frac{2}{3}$, find $\cos\theta$, $\tan\theta$, $\sin 2\theta$



$\cos\theta = \frac{\sqrt{5}}{3}$ $\tan\theta = \frac{2}{\sqrt{5}}$ $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{2}{3}\cdot\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$.

§2.5 Inverse functions

recall : $f: A \to B$ (domain, range), $x \mapsto f(x)$

want: the inverse function should be the reverse of this $f^{-1}: B \to A$, $f(x) \mapsto x$

problem : the inverse is often not a function

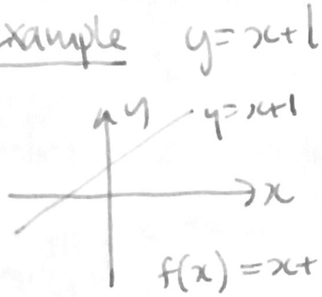
$f(a) = f(b)$ spose $a \neq b$ but $f(a) = f(b)$, what is $f^{-1}(f(a))$?

Q: when does a function $f: \mathbb{R} \to \mathbb{R}$ have an inverse?

A : when it passes the horizontal line line test (one-to-one / injective)

$\Leftrightarrow$ for each $c \in$ range, there is a unique $x \in$ domain s.t. $f(x) = c$.

Example $y = x+1$.



$y = x+1 \Leftrightarrow x = y-1$

$f^{-1}(x) = x-1$.

Q: how do we find a formula for the inverse?

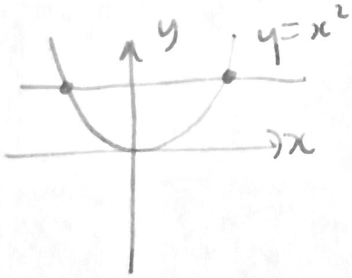
A : ① write down $y = f(x)$

② solve for x in terms of y, ie. $x = g(y)$

③ $f^{-1}(x) = g(x)$.

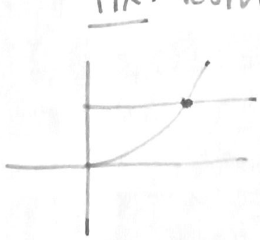
④ check!

bad example : $f(x) = x^2$



problem: doesn't pass horizontal line test

fix: restrict domain, consider $f: [0,\infty) \to [0,\infty)$



passes horizontal line test

so has an inverse we call $\sqrt{x}$.

$f^{-1}: [0,\infty) \to [0,\infty)$

$x \longmapsto \sqrt{x}$.