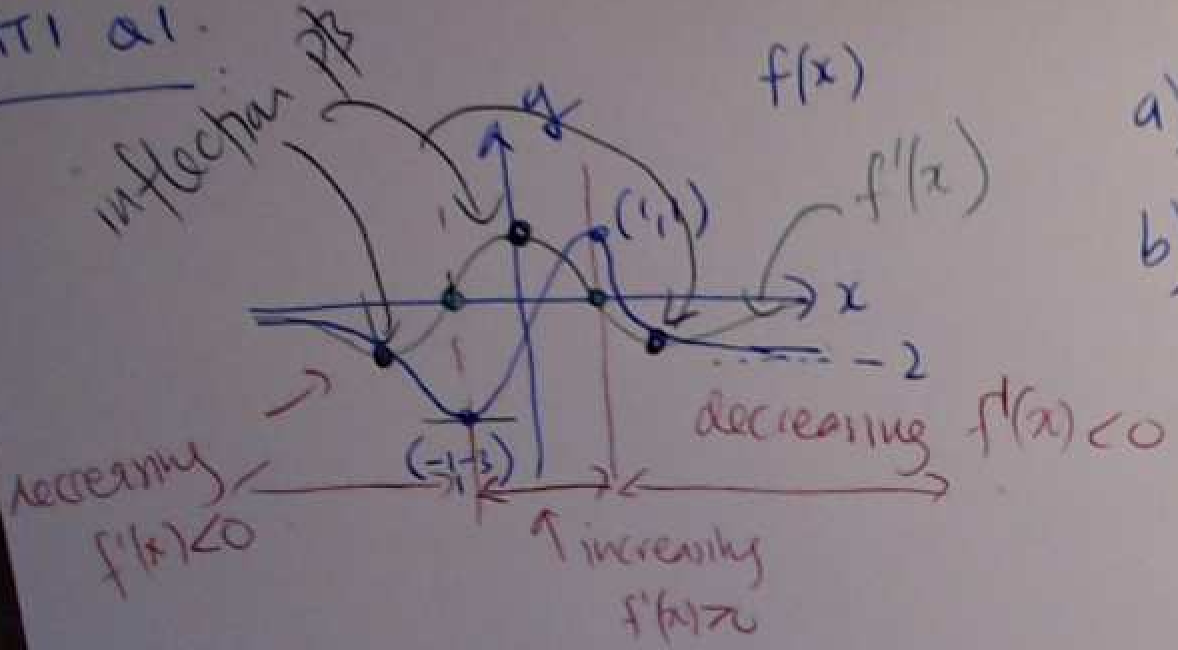
SMT1 Q1.



a) $f'(x) < 0 \leftrightarrow$ decreasing

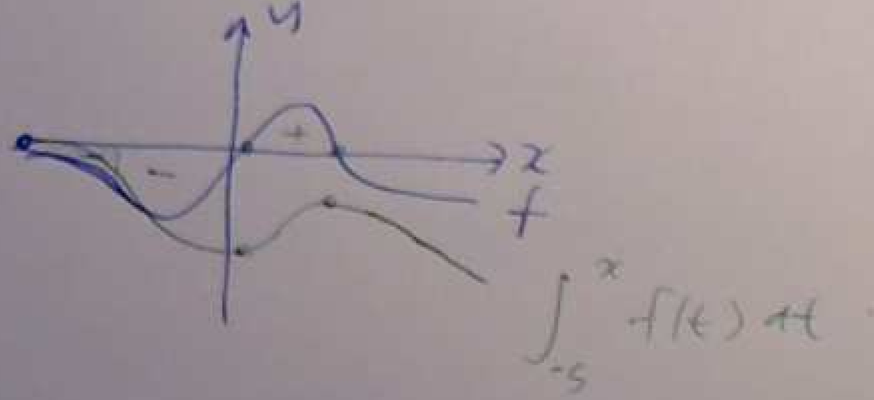
b) $f'(x) > 0 \leftrightarrow$ increasing

c) $\lim\limits_{x \to \infty} f'(x) = 0$.

d) $\lim\limits_{x \to -\infty} f'(x) = 0$.

e) sketch $f'(x)$

f) sketch $\int_{-5}^{x} f(t)\,dt$.

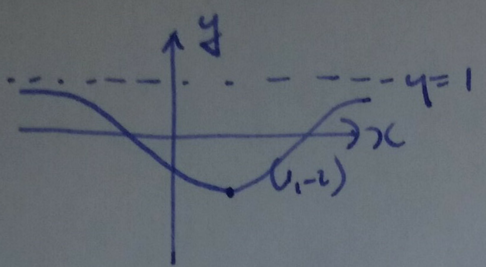


g) points of inflection

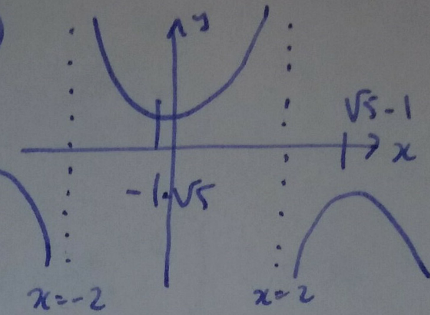
$f''(x) = 0$.

SMT 3 Solutions

Q2



e)



Q3 $f(x)=\dfrac{e^x}{(2-x)(2+x)}$ a) vertical asymptote $x=\pm 2$

b) horizontal asymptotes $\lim_{x\to\infty}\dfrac{e^x}{4-x^2}=\infty$

$\lim_{x\to-\infty}\dfrac{e^x}{4-x^2}=0$ e $y=0$.

$$f'(x)=\frac{(4-x^2)e^x-e^x(-2x)}{(4-x^2)^2}=\frac{e^x(4\overset{-2x}{\cancel{+}}x^2)}{(4-x^2)^2}$$ ~~so no critical points.~~

~~$f'(x)>0$ always increasing~~ critical points $x=\dfrac{-2\pm\sqrt{4+16}}{2}=-1\pm\sqrt{5}$.

increasing in $(-1-\sqrt{5},-1+\sqrt{5})$, decreasing outside.

$\cancel{f''(x)=\dfrac{(4-x^2)^2\cdot e^x(4-2x-x^2)-c^3}{(4-x^2)^4}}$ don't use 2nd derivative test - use first derivative test.

$-1-\sqrt{5}$ local min, $-1+\sqrt{5}$ local max.

Q4 $g(x)=x\ln x-2x$

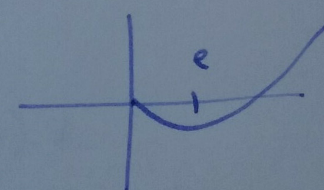
a) $g'(x)=\ln x+x\cdot\frac{1}{x}-2=\ln x-1$ $g'(x)=0$ $\ln x=1$ $x=e$

b) $f'(x)<0$ for $(0,e)$ $f'(x)>0$ for (e,∞). $\Rightarrow$ e local min.

c) $g''(x)=\frac{1}{x}\neq 0$ no points of inflection.

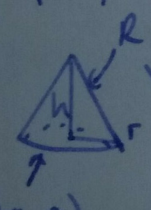
d) $\frac{1}{x}>0$ convex up.

e)

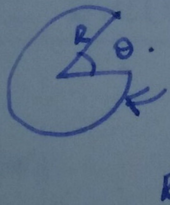


Q5 $f'(x)<0$ for all x, so decreasing, so max at $x=1$.

Q6



$V=\frac{1}{3}\pi r^2h$



$\cancel{2\pi}R(2\pi-\theta)$

$2\pi r=R(2\pi-\theta)$

$r=R-\dfrac{\theta R}{2\pi}$.

$R^2=h^2+r^2$

$h=\sqrt{R^2-r^2}$.

$$V=\frac{1}{3}\pi R^3\left(1-\frac{\theta}{2\pi}\right)^2\sqrt{1-\left(1-\frac{\theta}{2\pi}\right)^2}.$$

$$V=\frac{1}{3}\pi\left(R-\frac{\theta R}{2\pi}\right)^2\sqrt{R^2-\left(R-\frac{\theta R}{2\pi}\right)^2}$$

$$\cancel{V=\frac{1}{3}\pi\left(R-\frac{\theta R}{2\pi}\right)^2\sqrt{\frac{R^2\theta^2}{4\pi^2}-\frac{2R^2\theta}{2\pi}}}$$

$$V=\frac{1}{3}\pi R^3\left(1-\frac{\theta}{2\pi}\right)^2\sqrt{\frac{\theta^2}{4\pi^2}-\frac{\theta}{\pi}}$$

$$\frac{dV}{d\theta} = \frac{1}{3}\pi R^3 \cdot 2\left(1-\frac{\theta}{2\pi}\right)\left(-\frac{1}{2\pi}\right)\sqrt{} + \frac{1}{3}\pi R^3\left(1-\frac{\theta}{2\pi}\right)^2 \frac{1}{2}\left(1-\left(1-\frac{\theta}{2\pi}\right)^2\right)^{-1/2} \cdot -2\left(1-\frac{\theta}{2\pi}\right)\cdot\left(\frac{-1}{2\pi}\right) \quad ②$$

$$= \frac{\frac{1}{3}\pi R^3\left(-\frac{1}{2\pi}\right)}{\sqrt{1-(1-\theta/2\pi)^2}}\left[2\left(1-\frac{\theta}{2\pi}\right)\left(1-\left(1-\frac{\theta}{2\pi}\right)^2\right) \overset{-}{\cancel{4}} \left(1-\frac{\theta}{2\pi}\right)^3\right].$$

$$= \frac{\frac{1}{3}\pi R^3\left(-\frac{1}{2\pi}\right)}{\sqrt{1-(1-\theta/2\pi)^2}} \cdot \left(1-\frac{\theta}{2\pi}\right)\left[2\left(1-\left(1-\frac{\theta}{2\pi}\right)^2\right) \overset{-}{\cancel{4}} \left(1-\frac{\theta}{2\pi}\right)^2\right] = 0.$$

$$2 - 3\left(1-\frac{\theta}{2\pi}\right)^2 = 0 \qquad \left(1-\frac{\theta}{2\pi}\right)^2 = 2/3. \qquad 1-\frac{\theta}{2\pi} = \sqrt{2/3}.$$

$$\theta = 2\pi\left(1-\sqrt{2/3}\right).$$

Q7 a) $$\lim_{x\to-\infty} \frac{2-3x}{\sqrt{2x^2-3}} = \lim_{x\to+\infty} \frac{2+3x}{\sqrt{2x^2-3}} = \lim_{x\to+\infty} \frac{3+2/x}{\sqrt{2-3/x^2}} = \frac{3}{\sqrt{2}}.$$

b) $$\lim_{x\to 0} \frac{\sin^{-1}(2x)}{\cos^{-1}(3x)} = \frac{\frac{1}{\sqrt{1-(2x)^2}}\cdot 2}{\frac{-1}{\sqrt{1-(3x)^2}}\cdot 3}. = \frac{2}{3}.$$

c) $$\lim_{x\to 0} \sin(x)\ln(x) = \lim_{x\to 0} \frac{\ln(x)}{\operatorname{cosec}(x)} = \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec}x\cot x} = \lim_{x\to 0} -\frac{\tan x \sin x}{x}$$

$$= \lim_{x\to 0} \frac{\sec^2 x \sin x + \tan x \cos x}{1} = 0$$

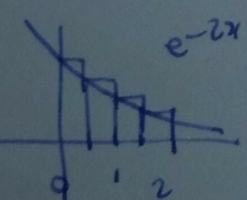
d) $$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{e^x-1}\right) = \lim_{x\to 0} \frac{e^x-1-\sin x}{\sin x(e^x-1)} = \lim_{x\to 0} \frac{e^x - \cos x}{\cos x(e^x-1)+\sin x\, e^x}$$

$$= \lim_{x\to 0} \frac{e^x+\sin x}{-\sin x(e^x-1)+2\cos x(e^x)+\sin x\, e^x} = \frac{1}{2e}$$

e) $$\lim_{x\to 0} \frac{3\tan x - \tan 3x}{\sin^2 2x} = \lim_{x\to 0} \frac{3\sec^2 x - \sec^2(3x).3.}{2\sin 2x.\cos 2x.2 = 2\sin 4x.}$$

$$= \lim_{x\to 0} \frac{6\sec x.\sec x\tan x - 6\sec(3x).\sec(3x)\tan(3x).3.}{\cancel{2}\,\overset{\cos}{\cancel{\sin}}4x.4} = \frac{0}{8} = 0.$$

Q8



are estimate

$$\frac{1}{2}\left(f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)\right) = \frac{1}{2}\left(e^{-0}+e^{-1}+e^{-2}+e^{-3}\right) \approx 0.777\ldots$$

Q9 a) $\int 2x^{-1/3} + 3x^{2/3} - 2x^{5/3}\,dx = \frac{6}{2}x^{2/3} + \frac{9x^{5/3}}{5} - \frac{6}{8}x^{8/3} + c.$

b) $\int_{-2}^{1} |x|\,dx = \int_{-2}^{0} -x\,dx + \int_{0}^{1} x\,dx = \left[-\frac{1}{2}x^2\right]_{-2}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{1} = \frac{1}{2}\cdot 4 + \frac{1}{2} = \frac{5}{2}$

c) $\int_{1}^{8} \frac{2}{x}\,dx = \left[2\ln|x|\right]_{1}^{8} = 2\ln 8$

d) $\int_{0}^{x} \frac{1}{t+3}\,dt = \left[\ln|t+3|\right]_{0}^{x} = \ln|x+3| - \ln 3.$

e) $\int \frac{1}{1+4x^2}\,dx \qquad u = 2x \quad \frac{du}{dx} = 2 \qquad \int \frac{1}{1+u^2}\frac{dx}{du}\,du = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u) + c$

$= \frac{1}{2}\tan^{-1}(2x) + c$

f) $\int \frac{x}{2+x^2}\,dx \qquad u = 2+x^2 \quad \frac{du}{dx} = 2x \qquad \int \frac{x}{u}\frac{dx}{du}\,du = \int \frac{x}{u}\cdot\frac{1}{2x}\,du = \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2+x^2| + c.$

g) $\int \frac{\sqrt{x}}{x+x^2}\,dx \qquad u = \sqrt{x} \quad \frac{du}{dx} = \frac{1}{2}x^{-1/2} \qquad \int \frac{\sqrt{x}}{u^2+u^4}\frac{dx}{du}\,du = \int \frac{u}{u^2+u^4}\,\frac{1}{\frac{1}{2}\frac{1}{u}}\,du$

$= 2\int \frac{u^2}{u^2+u^4}\,du = 2\int \frac{1}{1+u^2}\,du = 2\tan^{-1}(u) = 2\tan^{-1}(\sqrt{x}) + c.$

Q10 $v(t) = (t+1)^{-4}.$

$x(t) = \int (t+1)^{-4}\,dt = -\frac{1}{3}(t+1)^{-3} + c \qquad x(0) = 0 \Rightarrow c = \frac{1}{3}.$

$x(t) = \frac{1}{3} - \frac{1}{3}(t+1)^{-3} \to \frac{1}{3}$ as $t \to \infty$, doesn't reach 1.