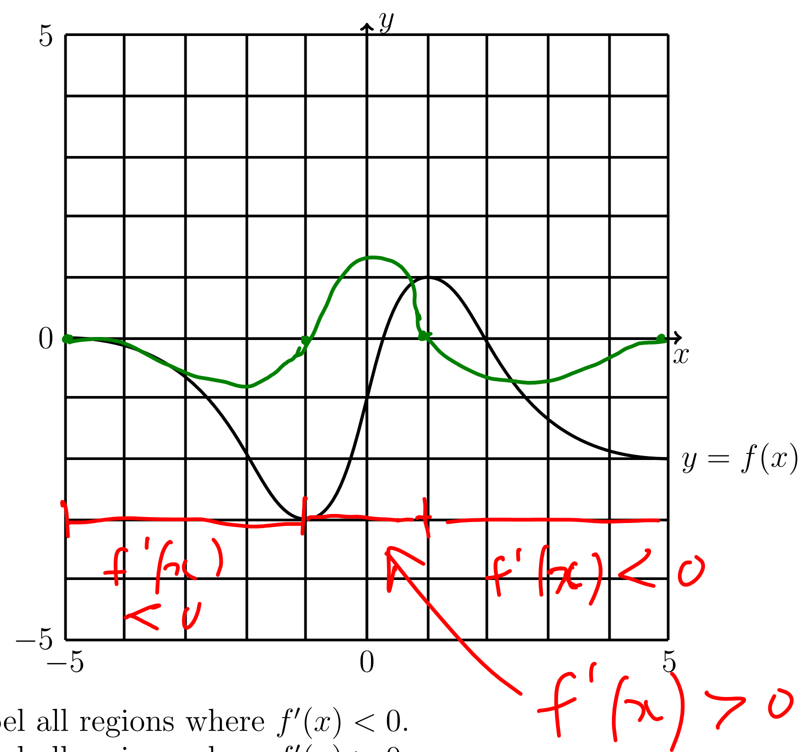
Math 231 Calculus 1 Fall 20 Sample Midterm 2

(1) Consider the function $f(x)$ defined by the following graph.



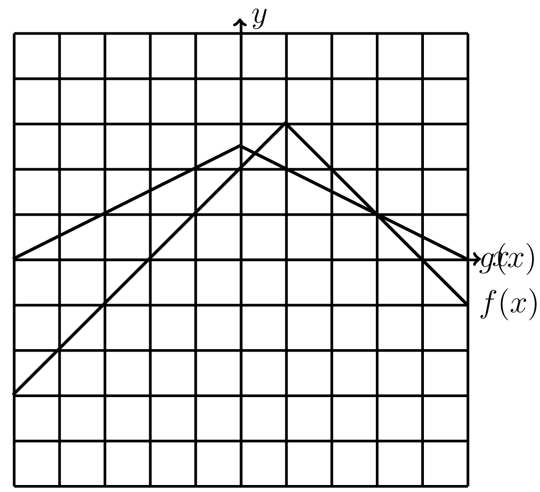
(a) Label all regions where $f'(x) < 0$.
(b) Label all regions where $f'(x) > 0$.
(c) Sketch a graph of $f'(x)$ on the figure.
(d) What is $\lim_{x\to\infty} f(x)$? -2
(e) What is $\lim_{x\to-\infty} f'(x)$? 0

(2) Find the derivatives of the following functions

(a) $x^5 e^{-3x^3}$
(b) $\dfrac{\sqrt{2x-1}}{3-\tan(2x)}$
(c) x^{4x}
(d) $\ln(\sec(\sqrt{x}))$
(e) $\tan^{-1}(2/\sqrt[4]{x})$
(f) $\sin^{-1}(3-2x)$

(3) Find the second derivatives of the functions above.

(4) The graphs of the functions f and g are shown below.



(a) Let $h(x) = f(x)g(x)$ Find $h'(3)$.
(b) Let $h(x) = f(g(x))$. Find $h'(-1)$.

(5) Use implicit differentiation to find the tangent line to the hyperbola $16x^2 - 3y^2 = 4$ at the point $(1, -2)$.

(6) Find $\frac{dy}{dx}$ for the implicit function $x^3y + x^2y^2 = \sin(xy)$.

(7) You inflate a spherical balloon at a rate of 20cm^3 per second. How fast is the area of the balloon increasing when the radius is 20cm?

(8) Use a linear approximation to estimate $\sqrt[3]{26}$. What is the percentage error?

(9) Find all the critical points for the function $f(x) = e^x(x^2 - x - 5)$. Use the first derivative test to identify them as local maxima or local minima.

(10) Find the absolute maximum and minimum of the function $f(x) = x^2 - 2x - 3$ on the interval $[-2, 2]$.

SMT2 Solutions

Q3 a) $f(x) = x^5 e^{-3x^3}$

$$f'(x) = 5x^4 e^{-3x^3} + x^5 e^{-3x^3} \cdot -9x^2 = 5x^4 e^{-3x^3} - 9x^7 e^{-3x^3}$$

$$f''(x) = (20x^3 - 63x^6)e^{-3x^3} + (5x^4 - 9x^7)e^{-3x^3} \cdot -9x^2 .$$

b). $f(x) = \dfrac{\sqrt{2x-1}}{3-\tan(2x)}$ $\quad f'(x) = \dfrac{(3-\tan(2x)) \cdot \frac{1}{2}(2x-1)^{-1/2} \cdot 2 - 2\sec^2(2x) \cdot \sqrt{2x-1}}{(3-\tan(2x))^2}$.

$$\cancel{f''(x)} = f'(x) = \frac{3-\tan(2x) - 2\sec^2(2x)\cdot(2x-1)}{(3-\tan(2x))^2\sqrt{2x-1}} .$$

$$f''(x) = \frac{(3-\tan(2x))^2\sqrt{2x-1}\left(-\sec^2(2x)\cdot 2 - 2\sec^2(2x)\cdot 2 - 4\sec(2x)\cdot\sec(2x)\tan(2x)\cdot 2(2x-1)\right) - ①}{(3-\tan(2x))^4(2x-1)} .$$

$$① = \left(3-\tan(2x) - 2\sec^2(x)(2x-1)\right) \cdot \left(2(3-\tan(2x)) \cdot \sec^2(2x) \cdot 2\sqrt{2x-1} + (3-\tan(2x))^2 \tfrac{1}{2}(2x-1)^{-1/2} \cdot 2 \right).$$

c) $x^{4x} = e^{4x\ln(x)}$.

$$f'(x) = e^{4x\ln(x)} \cdot \left(4\ln(x) + \frac{4x}{x}\right) = e^{4x\ln(x)}\left(4\ln(x) + 4\right).$$

$$f''(x) = e^{4x\ln(x)}\left(4\ln(x)+4\right)^2 + e^{4x\ln(x)} \cdot \left(\frac{4}{x}\right).$$

d) $\ln(\sec(\sqrt{x}))$ $\quad f'(x) = \dfrac{1}{\sec(\sqrt{x})} \sec(\sqrt{x})\tan(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} = \dfrac{1}{2}\dfrac{\tan(\sqrt{x})}{\sqrt{x}}$.

$$f''(x) = \frac{1}{2}\,\frac{\sqrt{x}\sec^2(\sqrt{x})\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-1/2}\tan(\sqrt{x})}{\cancel{4}x} .$$

e) $\tan^{-1}(2x^{-1/4})$ $\quad f'(x) = \dfrac{1}{1+4x^{-1/2}} \cdot 2 \cdot -\frac{1}{4}x^{-5/4} = \dfrac{-\frac{1}{2}x^{-5/4}}{1+4x^{-1/2}}$

$$f''(x) = \frac{(1+4x^{-1/2})\left(\frac{5}{8}x^{-9/4}\right) + \frac{1}{2}x^{-5/4} \cdot -2x^{-3/2}}{(1+4x^{-1/2})^2} .$$

f) $\sin^{-1}(3-2x)$. $f'(x) = \dfrac{1}{\sqrt{1-(3-2x)^2}} \cdot -2 = -2\left(1-(3-2x)^2\right)^{-1/2}$

$f''(x) = \left(1-(3-2x)^2\right)^{-3/2} \cdot 2(3-2x) \cdot -2$

Q4 $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

at $x=3$: $f'(3)g(3)+f(3)g'(3) = (-1)1 + 1 \cdot \frac{1}{2} = -\frac{3}{2}$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$ $\quad x=-1$: $f'(g(-1)) \cdot g'(-1)$.

$= f'(-2) \cdot \frac{1}{2} = 0$.

Q5 $16x^2 - 3y^2 = 4$ $\quad (1,-2)$

$32x - 6yy' = 0$ $\quad 32 + 12y' = 0$ $\quad y' = -\frac{8}{3}$ $\quad y+2 = -\frac{8}{3}(x-1)$.

Q6 $x^3y + x^2y^2 = \sin(xy)$

$3x^2y + x^3y' + 2xy^2 + x^2 2yy' = \cos(xy) \cdot (y + xy')$.

$y'(x^3 + 2x^2y - \cos(xy) \cdot x) = \cos(xy) \cdot y - 3x^2y - 2xy^2$.

$y' = \dfrac{y\cos(xy) - 3x^2y - 2xy^2}{x^3 + 2x^2y - \cos(xy) \cdot x}$.

Q7 $V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ $\quad \frac{dV}{dt} = 20$. $\quad r = 20$

$\frac{dr}{dt} = \dfrac{20}{4\pi \cdot 20^2} = \dfrac{1}{80\pi}$.

$A = 4\pi r^2$ $\quad \frac{dA}{dt} = 4\pi \cdot 2r \frac{dr}{dt} = \dfrac{4\pi \cdot 2 \cdot 20}{80\pi} = 2$.

Q8 $f(x) = \sqrt[3]{x} = x^{1/3}$ $\quad f(27) = 3$.

$f'(x) = \frac{1}{3}x^{-2/3} = \dfrac{1}{3\sqrt[3]{x^2}}$ $\quad f'(27) = \dfrac{1}{3 \cdot 9} = \dfrac{1}{27}$

$f(26) \approx f(27) - 1f'(27) = 3 - \frac{1}{27} = 2\frac{26}{27}$ $\quad$ percentage error $\dfrac{2\frac{26}{27} - \sqrt[3]{26}}{\sqrt[3]{26}} \times 100$

$\approx 0.016\%$.

Q9 $f(x) = e^x(x^2 - x - 5)$

$f'(x) = e^x(x^2 - x - 5) + e^x(2x-1) = e^x(x^2 + x - 6) = e^x(x-3)(x+2)$

solve $f'(x)=0$, critical points $x=3,-2$

$\cancel{f''(x) = e^x(x^2-x-6)+e^x}$

		-2		3	
e^x	$+$		$+$		$+$
$(x-3)$	$-$		$-$		$+$
$(x+2)$	$-$		$+$		$+$
$f'(x)$	$+$	-2 local max	$-$	3 local min.	$+$

Q10 $f(x) = x^2-2x-3$

$f'(x) = 2x-2$ $\quad f'(x)=0.\ x=1$ $\quad$ check $-2, 1, 2$

$f(-2) = 5$ ← abs max

$f(1) = -4$ ← abs min.

$f(2) = -3$