Sample final solutions

Q1 a) $-10x^4+2.\frac{4}{3}x^{1/3}+\sec^2(3x).3$

b) $f'(x)=\dfrac{\ln(2x+1).(2x-1)-(x^2-x).\frac{1}{2x+1}.2}{\ln(2x+1)^2}$

c) $f'(x)=-3e^{-3x}\cos(2-3x)+e^{-3x}.-\sin(2-3x).-3.$

d) $f'(x)=\frac{1}{4}\left(e^{-\sin(3x)}+2\right)^{-3/4}.e^{-\sin(3x)}.-\cos(3x).3$

Q2 a) $-x^{-1}+3\sin(x)-e^x+c$

b) $\displaystyle\int\frac{1-4x+4x^2}{x^{3/2}}dx=\int x^{-3/2}-4x^{-1/2}+4x^{1/2}dx=-2x^{-1/2}-4x^{1/2}.2+4.\frac{2}{3}x^{3/2}+c$

c) $\displaystyle\int_0^{\pi/4}\cos(4x)\sin^3(4x)\,dx$ $\qquad u=\sin(4x)\quad \frac{du}{dx}=\cos(4x).4$

$\displaystyle=\int\cos(4x).u^3\frac{dx}{du}du=\int\cos(4x)\,u^3.\frac{1}{\cos(4x).4}du=\frac{1}{4}\int u^3du=$

$\frac{1}{16}u^4+c=\frac{1}{16}\sin^4(4x)+c\qquad\left[\frac{1}{16}\sin(4x)\right]_0^{\pi/4}=0.$

d) $\displaystyle\int\frac{1}{1+4x^2}dx$ $\qquad u=2x\quad \frac{du}{dx}=2$ $\qquad\displaystyle\int\frac{1}{1+u^2}\frac{dx}{du}du=\int\frac{1}{1+u^2}.\frac{1}{2}du$

$=\frac{1}{2}\tan^{-1}(u)+c=\frac{1}{2}\tan^{-1}(2x)+c$

Q3 a) $\displaystyle\lim_{x\to3}\frac{x^2-x-6}{x-3}=\lim_{x\to3}\frac{(x-3)(x+2)}{x-3}=\lim_{x\to3}x+2=5$

b) $\displaystyle\lim_{x\to0}\frac{1-e^{3x}}{\cos(5x)}\;\cancel{\stackrel{l'H}{=}\lim_{x\to0}\frac{-3e^{3x}}{5\sin(5x)}}\;\frac{0}{1}=0.$

c) $\displaystyle\lim_{x\to0^+}x^{\sin(x)}=\lim_{x\to0^+}e^{\ln(x)\sin(x)}=e^{\lim_{x\to0^+}\ln(x)\sin(x)}$

$$\lim_{x\to 0+} \ln(x)\sin(x) = \lim_{x\to 0+} \frac{\ln(x)}{1/\sin x} \overset{l'H:}{=} \lim_{x\to 0+} \frac{1/x}{-\sin x^{-2}\cdot \cos x}$$

$$= \lim_{x\to 0+} \frac{\sin^2 x}{x\cos x} \overset{l'H:}{=} \lim_{x\to 0+} \frac{2\sin x\cos x}{\cos x + x\cdot -\sin x} = \frac{0}{1} = 0$$

so $\lim_{x\to 0+} x^{\sin x} = e^0 = 1$

d) $$\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{\sin^2 x} = \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2\sin^2 x} \overset{l'H:}{=} \lim_{x\to 0} \frac{\overbrace{2\sin x\cos x}^{\sin 2x} - 2x}{2x\sin^2 x + x^2\cdot \underbrace{2\sin x\cos x}_{\sin 2x}}$$

$$\overset{l'H:}{=} \lim_{x\to 0} \frac{\cos 2x\cdot 2 - 2}{2\sin^2 x + \underbrace{2x\cdot\underbrace{2\sin x\cos x}_{\sin 2x} + 2x\sin 2x}_{4x\sin 2x} + x^2\cdot 2\cos 2x\cdot 2}$$

$$\overset{l'H:}{=} \lim_{x\to 0} \frac{-4\sin 2x}{\underbrace{4\sin x\cos x + 4\sin 2x}_{6\sin 2x} + 4x\cos 2x\cdot 2 + 2x\cdot 4\cos 2x - x^2\, 8\sin 2x}$$

$$\overset{l'H}{=} \lim_{x\to 0} \frac{-8\cos 2x}{12\cos 2x - 8\cos 2x - 4x\cdot\sin 2x\cdot 4 + 8\cos 2x - 16x\sin 2x - 2x\cdot 8\sin 2x - x^2\cdot 16\cos 2x}$$

$$= \frac{-8}{12-8+8} = -\frac{2}{3}.$$

Q4 $f(x) = x^3 - 12x$ a) $f'(x) = 3x^2 - 12 = 0$ $3(x^2-4) = 0$ $x = \pm 2$ critical pts.

b)

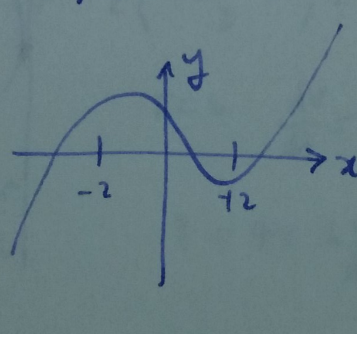
		-2		$+2$	
$x-2$	$-$		$-$		$+$
$x+2$	$-$		$+$		$+$
$f'(x)$	$+$		$-$		$+$

increasing on $(-\infty,-2)\cup(2,\infty)$

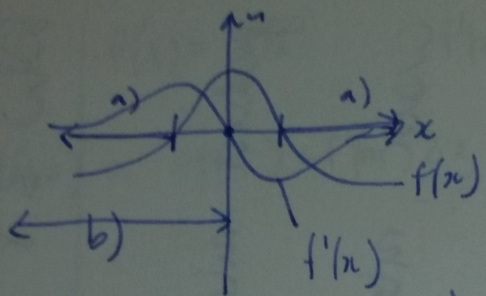
c) $f''(x) = 6x$ concave up $(0,\infty)$ concave down $(-\infty,0)$

d) -2 max ~~min~~ $+2$ min ~~max~~.

e)

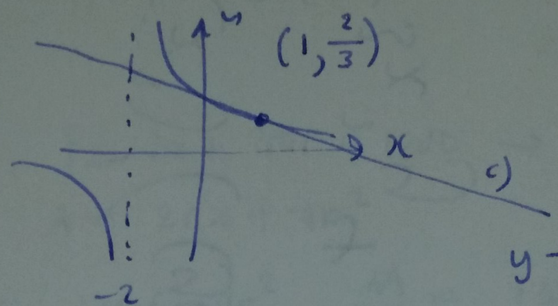


Q5



a) $f(x) < 0$

b) $f'(x) > 0$



$y = 0$ horizontal asymptote

Q6 $f(x) = \dfrac{2}{x+2}$

vertical asymptote at $x = -2$

$y - \frac{2}{3} = -\frac{2}{9}(x-1)$

$f'(x) = -2(x+2)^{-2} . 1 = \dfrac{-2}{(x+2)^2}$ b) $f'(1) = \dfrac{-2}{9}$

Q7 $f(x) = x^2 - x$ $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^2 - (x+h) - x^2 + x}{h}$

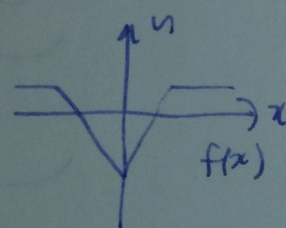
$= \lim_{h\to 0} \dfrac{\cancel{x^2} + 2xh + h^2 - \cancel{x} - h - \cancel{x^2} + \cancel{x}}{h} = \lim_{h\to 0} 2x + h - 1 = 2x - 1$

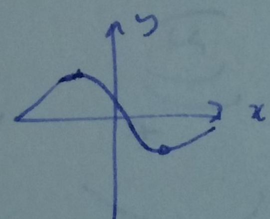
Q8 $x^3y - xy^4 + 4x = 10$ $(-1, 2)$

$3x^2 . y + x^3y' - y^4 - x . 4y^3y' + 4 = 0$ $6\,\cancel{4} - y' - 16 + 32y' + 4 = 0$

$y' = 6/31$ $y - 2 = \frac{6}{31}(x+1)$

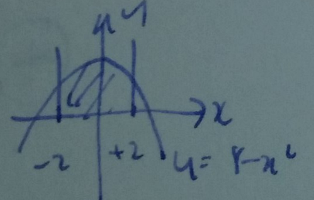
Q9





$\int_{-5}^{x} dx$.

Q10



$\int_{-2}^{2} 8 - x^2\,dx = \left[8x - \frac{1}{3}x^3\right]_{-2}^{2} = 16 - \frac{8}{3} - \left(-16 + \frac{8}{3}\right)$

$= 2\left(\dfrac{48 - 8}{3}\right) = \dfrac{80}{3}$

Q11 $V = \frac{4}{3}\pi r^3$ $A = 4\pi r^2$ $r = 6$ $\frac{dV}{dt} = 8$

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$ $\dfrac{dA}{dt} = 8\pi r \dfrac{dr}{dt}$ $8 = 4\pi . 36 \dfrac{dr}{dt}$ $\dfrac{dr}{dt} = \dfrac{1}{18\pi}$

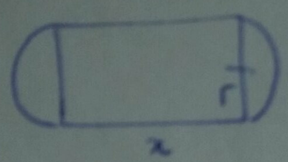
$\dfrac{dA}{dt} = \dfrac{8 . \pi . 6 .}{18\pi} = \dfrac{8}{3}\ in^2/s.$

Q12 $f(x) = \sqrt[3]{x}$ $\quad f'(x) = \frac{1}{3}x^{-2/3}$. $\quad 5^3 = 125$ so $f(125) = 5$.

$$f(120) \approx f(125) + f'(125)\underbrace{\Delta x}_{-5} = 5 + \frac{1}{3\cdot(125)^{2/3}} \cdot -5 = 5 + \frac{1}{3\cdot 25}\cdot -5$$

$$= 5 - \frac{1}{15} = 4\tfrac{14}{15}.$$

Q13.



$P = 2x + 2\pi r = 1200 \to 2x = 1200 - 2\pi r$

$A = 2rx + \pi r^2$

$A = r(1200 - 2\pi r) + \pi r^2 = 1200r - \pi r^2 \qquad \frac{dA}{dr} = 1200 - 2\pi r$

critical pt: $r = \frac{600}{\pi}$ $\qquad A = \frac{600}{\pi}\left(1200 - 2\pi\cdot\frac{600}{\pi}\right) + \pi\cdot\left(\frac{600}{\pi}\right)^2$

$= \frac{360000}{\pi}$ i.e. $x = 0$, circle.