Useful facts from probability   Law of Large numbers / CLT

Markov's inequality: r.v. $X \geq 0$ $\wedge$ $\mathbb{E}(X) < \infty$ then $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$

Proof $\mathbb{E}(X) \geq \mathbb{P}(X < a) \cdot 0 + \mathbb{P}(X \geq a) \cdot a$. $\square$

Corollary let $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be monotonically increasing. Then [illegible]

$$\mathbb{P}(X \geq a) = \mathbb{P}(\phi(X) \geq \phi(a)) \leq \frac{\mathbb{E}(\phi(X))}{\phi(a)} \quad \square.$$

Bernstein's estimate ~~inequality~~ X_i bounded r.v.s $S_n = X_1 + \cdots + X_n$

then $\mathbb{P}(|S_n - n\mathbb{E}(X_i)| \geq \epsilon n) \leq c^n$

Chernoff-Hoeffding estimates X_i exp. dist.

then $\mathbb{P}(S_n \geq (1+t) n \mathbb{E}(X_i)) \leq \left(\frac{1+t}{e^t}\right)^n$

~~Binom~~ Binomial dist: $X_i = \begin{cases} 1 & \text{w/prob } p \\ 0 & \text{w/prob } 1-p = q \end{cases}$ $\mathbb{E}(X_i) = p$, $\mathbb{E}(X_i^k) = p$

$$\mathbb{E}(e^{\lambda X_i}) = \mathbb{E}\left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} X^k\right) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathbb{E}(X^k)$$

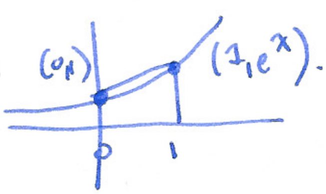
X_i binomial: $\mathbb{E}(e^{\lambda X_i}) = \underbrace{1}_{q+p} + \lambda p + \frac{\lambda^2}{2!} p + \frac{\lambda^3}{3!} p + \cdots = q + pe^{\lambda}$.
$= pe^{\lambda} + q$.

Set $S_n = X_1 + \cdots + X_n$ X_i indep. binom.

$$\mathbb{E}(e^{\lambda S_n}) = \mathbb{E}(e^{\lambda(X_1 + \cdots + X_n)}) = \prod_i e^{\lambda X_i} = (pe^{\lambda} + q)^n$$

More generally X_i bounded, takes values in $[0,1]$.

$e^{\lambda x}$ convex



$y - 1 = (e^{\lambda} - 1)x$

$y = (e^{\lambda} - 1)x + 1$.

$e^{\lambda x} \leq (e^{\lambda} - 1)x + 1$.

$\mathbb{E}(e^{\lambda X_i}) \leq \mathbb{E}((e^{\lambda} - 1)X + 1)$

$\leq (e^{\lambda} - 1)\mathbb{E}(X) + 1$

$\leq e^{\lambda} \underbrace{\mathbb{E}(X)}_{p} + 1 - \underbrace{\mathbb{E}(X)}_{q}$

(2e)

$\mathbb{E}(e^{\lambda S_n}) \leqslant (pe^{\lambda}+q)^n$ for $S_n = X_1 + \cdots + X_n$ $\quad X_i$ i.i.d. dist ~~bern~~ take values in $[0,1]$ w/ $\mathbb{E}(X_i) = p$.

Recall Markov: $\mathbb{P}(X \geqslant m) \leqslant \dfrac{\mathbb{E}(\phi(X))}{\phi(m)}$

so $\quad \mathbb{P}(S_n \geqslant m) \leqslant \dfrac{\mathbb{E}(e^{\lambda S_n})}{e^{\lambda m}} \leqslant \dfrac{(pe^{\lambda}+q)^n}{e^{\lambda m}}$ $\quad$ set $m = (p+t)n$

$$\mathbb{P}(S_n \geqslant (p+t)n) \leqslant \frac{(pe^{\lambda}+q)^n}{\left(e^{\lambda(p+t)}\right)^n}$$

$\Rightarrow$ Bernstein estimate: $\quad \mathbb{P}(|S_n - \underset{\mathbb{E}(X_i)}{\underset{\shortparallel}{p}}n| \geqslant \cancel{t}\epsilon n) \leqslant \left(\dfrac{pe^{\lambda}+q}{e^{\lambda(p+t)}}\right)^n$ $\quad \square$.

2. Linear progress w/ exp decay

Markov's inequality: $X \geq 0$ r.v. $\mathbb{E}(X) < \infty$. then $\mathbb{P}(X \geq a) \leq \dfrac{\mathbb{E}(X)}{a}$

Proof $\mathbb{E}(X) \geq \mathbb{P}(X \leq a) \cdot 0 + \mathbb{P}(X \geq a) \cdot a$ $\square$.

Corollary let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a monotonically increasing function. then

$$\mathbb{P}(|X| \geq a) = \mathbb{P}(\phi(|X|) \geq \phi(a)) \leq \frac{\mathbb{E}(\phi(|X|))}{\phi(a)} \quad \square.$$

Chernoff-Hoeffding bounds Z_i sequence of id. dist. ind. exp. r.v.s. Then $\forall t \geq 0$

$$\mathbb{P}(Z_1 + \cdots + Z_n \geq (1+t) n \mathbb{E}(Z_i)) \leq \left(\frac{1+t}{e^t}\right)^n.$$

Proof Let Z_i have pdf $f(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ $\quad \mathbb{E}(Z_i) = \frac{1}{\alpha}$. $\quad S_n = Z_1 + \cdots + Z_n$

moment generating function $\mathbb{E}(e^{\lambda Z_i}) = \mathbb{E}(1 + \lambda Z_i + \frac{\lambda^2}{2!} Z_i^2 + \cdots) = 1 + \lambda \mathbb{E}(Z_i) + \frac{\lambda^2}{2!} \mathbb{E}(Z_i)^2 + \cdots$

$$\mathbb{E}(e^{\lambda Z_i}) = \int_0^\infty \alpha e^{-\alpha x} \cdot e^{\lambda x} \, dx = \frac{\alpha}{-\alpha + \lambda} \left[e^{(-\alpha + \lambda)x} \right]_0^\infty = \frac{\alpha}{\alpha - \lambda} \quad (0 < \lambda < \alpha)$$

$$\mathbb{E}(e^{\lambda S_n}) = \mathbb{E}(e^{\lambda(Z_1 + \cdots + Z_n)}) = \mathbb{E}(e^{\lambda Z_1} \cdot e^{\lambda Z_2} \cdots e^{\lambda Z_n}) = \mathbb{E}(e^{\lambda Z_1}) \cdots \mathbb{E}(e^{\lambda Z_n}) = \left(\frac{\alpha}{\alpha - \lambda}\right)^n.$$

Markov's inequality: $\mathbb{P}(S_n \geq s) \leq \dfrac{\mathbb{E}(e^{\lambda S_n})}{e^{\lambda s}} = \dfrac{1}{e^{\lambda s}(1 - \lambda/\alpha)^n}$ $\circledast$

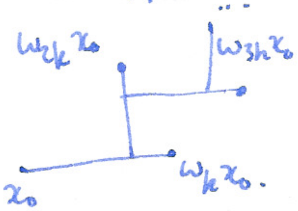
fact: $\circledast$ minimized by $\lambda = \alpha - \frac{n}{s}$.

so $\mathbb{P}(S_n \geq s) \leq \left(\frac{\alpha s}{n}\right)^n e^{-\alpha s + n}$, we choose $s = (1+t) n \mathbb{E}(Z_i) = (1+t) n \frac{1}{\alpha}$.

then $\mathbb{P}(S_n \geq (1+t) n \mathbb{E}(Z_i)) \leq \left(\dfrac{1+t}{e^t}\right)^n$ $\square$.

Thm $(G, \mu) \curvearrowright X$, μ non-elementary, bounded support in X. Then $\exists\, l, K_c$ s.t.

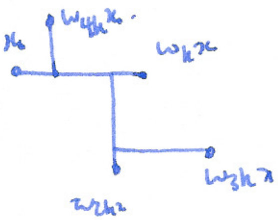
$$\mathbb{P}(d_X(x_0, w_n x_0) \leq ln) \leq K_c^n$$



Proof take k-steps at a time.

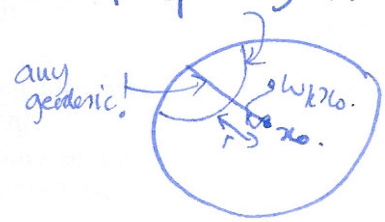
note: $(w_n x_0) \to \lambda((w_n)) \in \partial X \Rightarrow d_X(x_0, w_n x) \to \infty \Rightarrow \mathbb{E}(d_X(x_0, w_n x)) \to \infty$.

$$d_X(x_0, w_{nk} x_0) = d_X(x_0, w_k x) + d_X(w_k x_0, w_{2k} x_0) + \cdots + d_X(w_{(n-1)k} x_0, w_{nk} x_0).$$



$$-2(x_0, w_{2k} x_0)_{w_k x_0} - 2(x_0, w_{3k} x_0)_{w_{2k} x_0} \cdots$$

$$\mathbb{P}\left((x_0, w_{nk} x_0)_{w_{(n-1)k} x_0} \geq r\right) \leq \mu_k(\text{shadow of depth } r) \leq Kc^r$$



set $X_i^k = d(1, w_{ki} x_0) - d(1, w_{k(i-1)} x_0)$

$$X_i^k = Y_i^k - Z_i^k, \qquad Z_i^k = 2(1, w_{ki} x_0)_{w_{k(i-1)} x_0}.$$

$$Y_i^k = d_X(w_{(i-1)k} x_0, w_{ik} x_0)$$

Bernstein estimate $\mathbb{P}\left(\left|\sum_{i=1}^{n} (Y_i - \mathbb{E}(Y))\right| \geq \epsilon n\right) \leq c^n$.

Chernoff-Hoeffding estimate $\mathbb{P}\left(\sum Z_i^k \geq (1+t) n \mathbb{E}(Z_i)\right) \leq \left(\frac{1+t}{e^t \cancel{E}}\right)^n$ ← no k in here!

for large k $\mathbb{E}(Y) >> \mathbb{E}(Z)$. □

<u>Matching</u>: $stab_K(x) = \{g \in G \mid d_X(x, gx) \le K\}$.

$G \curvearrowright X$ <u>acylindrical</u>: $\forall K \ge 0\ \exists N, R$ s.t. $\forall x, y \in X$, $d_X(x,y) \ge R$, $|stab_K(x) \cap stab_K(y)| \le N$.

$G \curvearrowright X$, $g \in G$ is <u>WPD</u> if $\forall K \ge 0, \forall x \in X, \exists N$ s.t. $|stab_K(x) \cap stab_K(g^N x)| < \infty$.

<u>Intuition</u> from $F_2 \curvearrowright Cay(F_2)$ pick reduced word $w \in F_2$, take sample path γ, $\mathbb{P}(w$ matches at $\gamma(t)) \approx 3^{-|w|}$. ⇝ gives both matching and non-matching!

<u>Def</u>: $G \curvearrowright X$, we say a geodesic η <u>K-matches</u> γ if $\exists g$ s.t. $g\eta \subset N_K(\gamma)$
γ at $\gamma(t)$ if $g\eta \subset N_K([\gamma(t), \gamma(t+|\eta|)])$

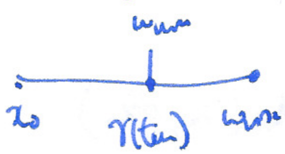
<u>Def</u>: α, β, (K,L)-match

<u>Example</u> $F_2 \curvearrowright \mathbb{H}^2$ dense image: everything matches! (not WPD, acylindrical).

some things don't match, e.g. a^n doesn't match b^n in F_2.

<u>Thm</u> [MT] $(G,\mu) \curvearrowright X$, supp$(\mu)$ bounded in X, g WPD

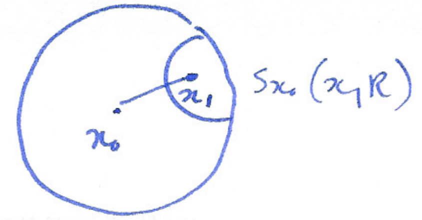
- matching: given (K,L) $\exists (B,c)$ s.t. $\mathbb{P}(\gamma_n = [x_0, w_n x_0]$ has a (K,L)-match w/ $\alpha_g) \ge 1 - Bc^{\sqrt{n}}$
- non-matching: given K, $\exists (B,c)$ s.t. $\forall m \le n$, $\mathbb{P}(\eta$ K-matches $[x_0, w_n x_0]$ at $\gamma(t_m)) \le Bc^{\sqrt{n}}$



<u>Applications</u>: "random small cancellation conditions".

useful facts

① Exponential decay for shadows

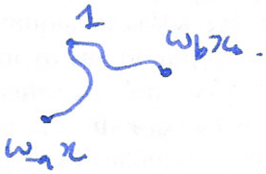


recall $S_{x_0}(x_1,R) = \{y \in X \mid (x_1, y)_{x_0} \geq d_X(x_0,x) - R\}$

Prop^n $(G,\mu) \curvearrowright X$ hyp μ non-elementary, supp(μ) bounded in X. Then there are constants K, c s.t. $\nu(S_{x_0}(x,R)) \leq K c^{d_X(x_0,x)-R}$ and $\mu_n(S_{x_0}(x,R)) \leq K c^{d_X(x_0,x)-R}$

Corollary (Gromov product bounds) $(G,\mu) \curvearrowright X$ hyp, as above. $\mathbb{P}((x_0, w_n x_0)_{w_i x_0} \geq r) \leq K c^r$

Proof



large Gromov product $\Rightarrow w_b x \in S_x(w_a x, R)$. □.

② Linear progress w/ exp decay

Thm [Mathieu-Sisto] (cylindrical) [Sunderland] (general case) $(G,\mu) \curvearrowright \delta X$, μ non-elementary, ~~supp~~ exp tail, (finite exp moment) i.e. $\exists \lambda$ s.t. $\mathbb{E}(e^{\lambda d_X(x_0, g x_0)}) < \infty$, then $\exists \ell > 0$, K, c $\mathbb{P}(|\frac{1}{n} d_X(x_0, w_n x) - \ell n| < \epsilon) \leq K c^n$.

Tools from probability

- Markov's inequality X r.v. $X \geq 0$ then $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$.

Proof $\mathbb{E}(X) \geq \mathbb{P}(X < a) \cdot 0 + \mathbb{P}(X \geq a) \cdot a$. □.

Corollary $\phi : \mathbb{R} \to \mathbb{R}$ monotonically increasing, then $\mathbb{P}(X \geq a) = \mathbb{P}(\phi(X) \geq \phi(a)) \leq \frac{\mathbb{E}(\phi(X))}{\phi(a)}$ □

- Bernstein estimates X_i bounded r.v.s. $S_n = X_1 + \cdots + X_n$ indep id. dist.

then $\exists c$ s.t. $\mathbb{P}(|S_n - n\mathbb{E}(X_1)| \geq \epsilon n) \leq c^n$

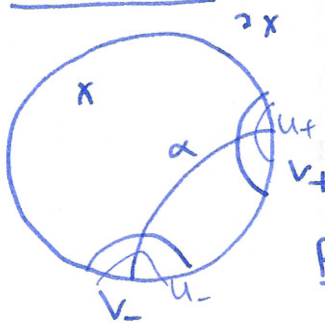
- Chernoff-Hoeffding estimates X_i finite exp moment, indep. id dist., then

$\mathbb{P}(S_n \geq (1+t) n \mathbb{E}(X_1)) \leq \left(\frac{1+t}{e^t}\right)^n$

• **matching (sketch)**

Thm $(G,\mu) \curvearrowright X$ μ non-elementary, bounded supp. in X. $\mathbb{P}([x_0, w_n x_0]$ has (K,L)-match w/ $\alpha) \geq 1 - Bc^{\sqrt{n}}$.

Pf (sketch). ergodicity $\Rightarrow \mathbb{P} \to 1$ but no rate.



Prop^n $A = \{ \omega \in (G,\mu)^{\mathbb{Z}}$ s.t. $(\partial_+(\omega), \partial_-(\omega)) \in V_+ \times V_- \}$.

$\exists B, c$ s.t. $\mu^{\mathbb{Z}}(A \cup T^{-1}A \cup \cdots \cup T^{-n}A) \geq 1 - Bc^{\sqrt{n}}$.

Proof (sketch) suffices to show $\mu^{\mathbb{Z}}(G \setminus (A \cup T^{-1}A \cup \cdots \cup T^{-n}A)) \leq Bc^{\sqrt{n}}$.

$= \mu^{\mathbb{Z}}(A^c \cap T^{-1}A^c \cap \cdots \cap T^{-n}A^c) \leq Bc^{\sqrt{n}}$.

set $A_n = A^c \cap T^{-1}A^c \cap \cdots \cap T^{-n}A^c$

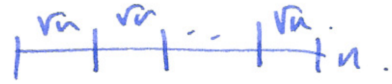
approx $\partial^{-1}(V_+ \times V_-)$ by $U_k = \{ \omega \in G^{\mathbb{Z}} \mid (w_k(\omega)x_0, w_{-k}(\omega)x_0) \in \cancel{V_+ \times V_-}\ U_+ \times U_- \}$ ← only depends on $[-k, k]$

set $V_k = \left\{ \omega \in G^{\mathbb{Z}} \;\middle|\; (w_n(\omega)x_0, w_{-n}(\omega)x_0) \begin{array}{ll} \in U_+ \times U_- & \text{for } n = k \\ \in V_+ \times V_- & \text{for } n \geq k \end{array} \right\}$.

$\partial(V_k) \subseteq V_+ \times V_-$, so $V_k \subseteq A$, so $A^c \subseteq V_k^c$

so $A_n \subseteq V_k^c \cap T^{-1}V_k^c \cap \cdots \cap T^{-n}V_k = \bigcap_{i=0}^{n} T^{-i}V_k^c$

pass to approx $\sqrt{n}$ sets of gap size $\sqrt{n}$.



$$A_n \subseteq \bigcap_{i=0}^{\sqrt{n}} T^{-\sqrt{n}i} V_{\sqrt{n}}^c.$$

$$V_k \subseteq U_k \Rightarrow V_k^c = U_k^c \cup (U_k \setminus V_k).$$

$$A_n \subseteq \bigcap_{i=0}^{\sqrt{n}} T^{-\sqrt{n}i}(U_k^c \cup (U_k \setminus V_k)).$$

$$A_n \subseteq \Big(\underbrace{\bigcap_{i=0}^{\sqrt{n}} T^{-\sqrt{n}i} U_k^c}_{\uparrow \text{independent!}} \Big) \cup \Big(\bigcup_{i=0}^{\sqrt{n}} T^{-\sqrt{n}i} (U_{\sqrt{n}} \setminus V_{\sqrt{n}}) \Big).$$

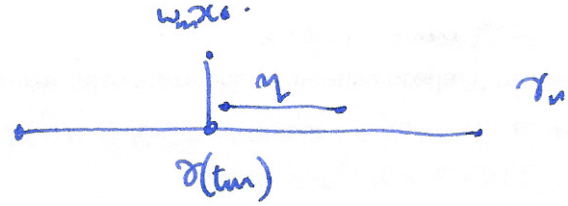
$$\mu^{\mathbb{Z}}(A_n) \leq \mu^{\mathbb{Z}}(U_k^c)^{\sqrt{n}} + \sqrt{n}\, \underbrace{\mu^{\mathbb{Z}}(\cancel{V_k^c \setminus U_k^c}\ U_{\sqrt{n}} \setminus V_{\sqrt{n}})}_{\text{shadow decay } \leq B_4 c^{4\sqrt{n}}};$$

$\longrightarrow \nu(U_k^c) < 1$. $\square$.

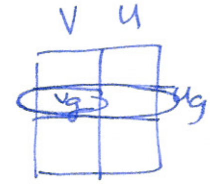
- non-matching

Thm $(G,\mu) \curvearrowright X$ μ non-elementary, bounded support in X, $\mathbb{P}([x_0, w_n x_0] = \gamma_n \ (K)\text{-matches } \eta \text{ at } \gamma_n(t_m)) \leq K c^{|\eta|}$.

Proof (sketch)



$|\eta| = 2s$



$U \subset (G,\mu)^{\mathbb{Z}}$ event that $\exists g$ s.t. $g\eta \subset N_K(\gamma(t_m), \gamma(t_m + 2s))$.

$V \subset (G,\mu)^{\mathbb{Z}}$ $\exists g$ s.t. $g\eta([0,s]) \subset N_K(\gamma(t_m), \gamma(t_m + s))$ $\quad V \subseteq U$.

conditional prob $\mathbb{P}(U) \leq \mathbb{P}(U|V)$.

$U_g \subseteq U$: some specific $g\eta \subset N_K(\gamma(t_m), \gamma(t_m + 2s))$ $\quad U = \bigcup_g U_g$

V_g first half $\quad V = \bigcup_g V_g$

exponential decay of shadows $\Rightarrow$ $\mathbb{P}(U_g | V_g) \leq Bc^s$

key fact for $|\eta| \gg 0$, any point of V is contained in a bounded number $\leq N(K)$ of sets V_g.

so $\mathbb{P}(U|V) \leq N(K) \cdot Bc^s$. $\square$