Notes on ergodic theory

(X,μ) finite measure space, wlog $\mu(X) = 1$.

$T \curvearrowright (X,\mu)$ measure preserving transformation, i.e. $\mu(T^{-1}A) = \mu(A)$ for all measurable $A \subseteq X$.

Def^n $T \curvearrowright (X,\mu)$ is ergodic if all T-invariant sets either have measure 0, or their complements have measure 0.

Prop^n the shift map $T \curvearrowright (G,\mu)^{\mathbb{Z}}$ is ergodic. $T(g_n) = (g_{n+1})$

Def^n T is mixing if for all measurable sets $A, B \subseteq X$, $\mu(T^{-k}A \cap B) \to \mu(A)\mu(B)$ as $k \to \infty$.

Prop^n mixing $\Rightarrow$ ergodic
$\not\Leftarrow$

Proof suppose $T \curvearrowright (X,\mu)$ not ergodic, then $\exists\, A \subseteq X$ with $T^{-1}A = A$ and $0 < \mu(A) < 1$

consider $\mu(T^{-k}A \cap A) \xrightarrow{\text{mixing}} \mu(A)\mu(A) = \mu(A)^2$ as $k \to \infty$.

$\quad \| $

$\mu(A \cap A) = \mu(A)$ so $\mu(A) = \mu(A)^2 \Rightarrow \mu(A) = 0$ or 1. $\square$

useful fact: $A \subseteq (G,\mu)^{\mathbb{Z}}$ can be approximated by cylinder sets, in fact given $(g_n) \in A$ set $B_n = \cdots G \times \{g_{-n}\} \times \cdots \times \{g_n\} \times G \times \cdots$ and $A_n = A \cup B_n$.
then $\mu(A \Delta A_n) \to 0$ as $n \to \infty$.

Prop^n $T \curvearrowright (G,\mu)^{\mathbb{Z}}$ is mixing

Proof $A, B \subseteq G^{\mathbb{Z}}$, $A_n \to A$, $B_n \to B$.

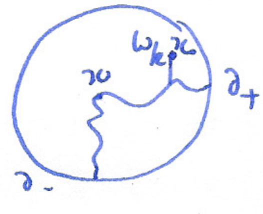
then $\mu(T^{-k}A_n \cap B_n) = \mu(A_n)\mu(B_n)$ for $k \gg n$.

$\therefore\ \mu(T^{-k}A \cap B) \to \mu(A)\mu(B)$ as $\square$.

Corollary any T-invariant property is the same for a.e. $(g_n) \in (G,\mu)^{\mathbb{Z}}$.

Q: how does T act? $T(g_n) = (g_{n+1})$, so $T^k(g_n) = (g_{n+k})$

$T \curvearrowright (G^{\mathbb{Z}}, \mathbb{P})$ path space as $T^k(w_n) = (w_k^{-1} w_n)$.



Poisson boundary

$(G,\mu)^{\mathbb{Z}} \to (G^{\mathbb{Z}}, \mathbb{P}) \leftarrow$ say two elements are tail equivalent if $\exists\, k, l, N$ s.t. $T^k(xg_n) = T^l(xg'_n)$ $\forall n \geq N$.

this gives an equivalence relation on a measure space, but this equivalence need not respect the Σ-algebra of the measure, however there is a Σ-algebra on $G^{\mathbb{Z}}/_\sim$ obtained by: $A \subseteq G^{\mathbb{Z}}/_\sim$ is measurable if inverse image in $G^{\mathbb{Z}}$ is measurable, so $(G^{\mathbb{Z}}/_\sim, \mathbb{P}/_\sim)$ inherits a measure from $(G^{\mathbb{Z}}, \mathbb{P})$.

Defn $(G^{\mathbb{Z}}, \mathbb{P})/_\sim$ is the Poisson boundary.

Prop^{n} For $(\mathbb{Z},\mu)$ simple r.w. $\mu(1) = 1/2$, $\mu(-1) = 1/2$, $(\mathbb{Z}^{\mathbb{Z}}, \mathbb{P})/_\sim$ is ~~a~~ $(\{pt\}$, ~~trivial~~ measure $1)$.

Proof Let A, B be cylinder sets in $(\mathbb{Z},\mu)^{\mathbb{Z}}$.

$\cdots\ \mathbb{Z}\times A \times \mathbb{Z} \times \cdots$ A

$\cdots\ \mathbb{Z}\times B \times \mathbb{Z} \times \cdots$ B.

then A / B contains all paths starting at $a \in A$ / $b \in B$ at time t_a / t_b

r.w. on $\mathbb{Z}$ recurrent, so these eventually hit 0, so up to shift T^k, contain same forward path, so $\cancel{A/_\sim \cap B/_\sim \neq \emptyset}$. $\exists\, \alpha \in A, \beta \in B$ s.t. $\alpha \sim \beta$, i.e. $\tilde{A} \cap \tilde{B}$ for all cylinder sets. $\Rightarrow (\mathbb{Z}^{\mathbb{Z}}, \mathbb{P})/_\sim = \{pt\}$. $\square$.

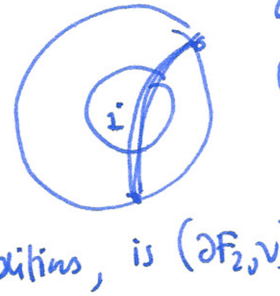
Intuition: Poisson boundary is "largest" boundary, contains all probabilistically nonzero behaviour.

Example (non-Poisson-boundary) $F_2 \times F_2 \curvearrowright Cay(F_2) \times Cay(F_2) \xrightarrow{\pi} Cay(F_2)$.

Thm [MT] $(G,\mu) \curvearrowright X$ hyp $\langle supp(\mu)\rangle_+$ non-elementary action acylindrical (or WPD) then $(\partial X, \nu)$ is a geometric realization of the Poisson boundary. (+moment conditions).

Tools [Kaimanovich] (ray/strip criterion).

strip criterion for $G \curvearrowright Cay(G)$ Gromov hyp



given $\alpha, \beta \in \partial X$ let $\sigma(\alpha,\beta) = \bigcup$ all geodesics from α to β.

show growth subexponential...

Open problem: $(F_2, \mu) \curvearrowright Cay(F_2)$, no moment conditions, is $(\partial F_2, \nu)$ Poisson boundary?