Continued fractions

Examples $\frac{4}{7} = 1 + \cfrac{1}{1+\cfrac{1}{3}}$ $[0;1,1,3]$

$\sqrt{2} \approx 1.4\ldots$ $\sqrt{2} = 1 + (\sqrt{2}-1)$ $\frac{1}{(\sqrt{2}-1)}\frac{\sqrt{2}+1}{(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{1} = 2 + (\sqrt{2}-1)$

$\sqrt{2} = 1 + \frac{1}{2+(\sqrt{2}-1)} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2\ldots}}}$ $[1;2,2,2,\ldots]$.

golden ratio : $\tau = \frac{1+\sqrt{5}}{2}$ $[1;1,1,1,\ldots]$.

e : $[2;1,2,1,1,4,1,1,6,1,1,8,\ldots]$.

π : $[3;7,15,1,292,1,1,1,2,1,3,1,\ldots]$

$\sqrt[3]{2}$: $[1;3,1,5,1,1,4,1,1,8,1,14,1,10,\ldots]$

Facts. $x \in \mathbb{Q} \Leftrightarrow$ finite

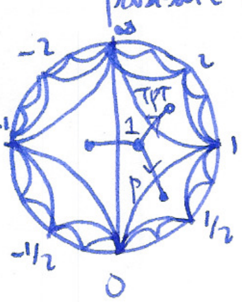
$x \in$ quadratic number field $\Leftrightarrow$ periodic

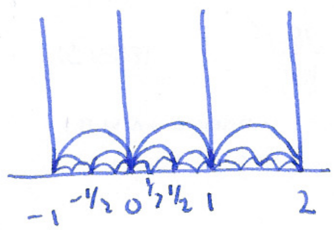
Algorithm : start with $x \in \mathbb{R}_{\geq 0}$ if $x > 1$ $x \mapsto x-1$; if $x < 1$ $x \mapsto \frac{1}{x}$. } dense orbit acting on $\mathbb{R}$.

action on $\mathbb{C}$: $z \mapsto z+1$, $z \mapsto 1/z$. } preserve $\mathbb{R}$, so give elements of $(P)SL(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$

$z \mapsto z+1 \leftrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ parabolic about 0

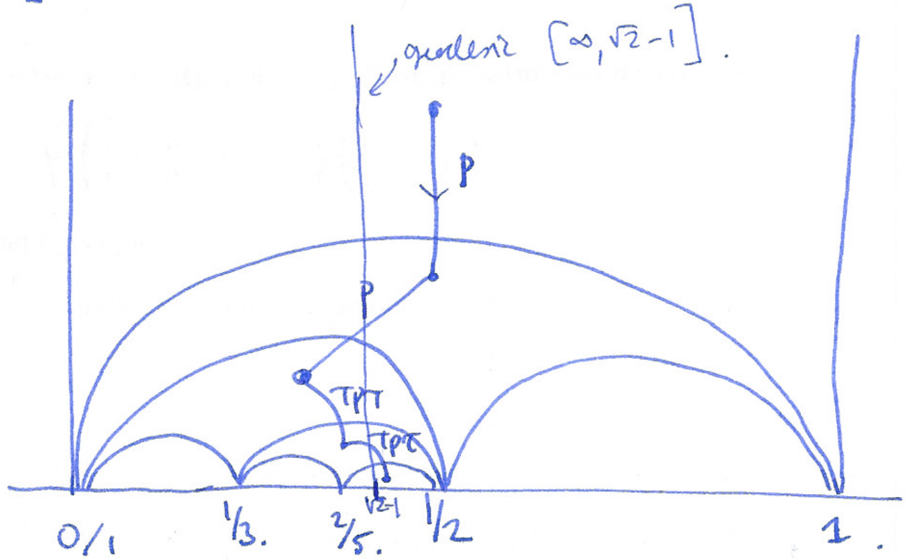
$z \mapsto \frac{1}{z} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflection in $[-1,1]$. ← these isometries preserve the Farey triangulation.





right turns about 0 : p^n

left turns about ∞ : $(\tau p \tau)^n = \tau p^n \tau$.



$\sqrt{2}-1 = [0;2,2,2,\ldots]$.

$\approx 0.414\ldots$

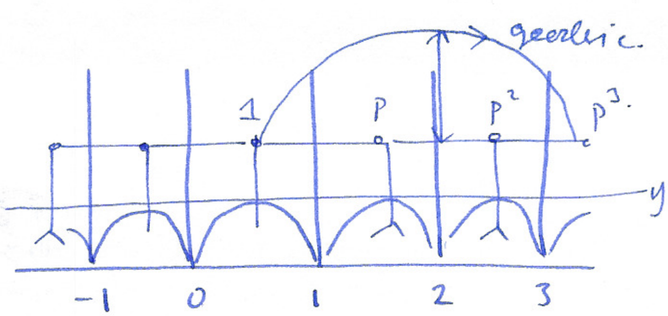
$\frac{1}{\sqrt{2}-1} \approx 2.414\ldots$

p^2 ↑ # right turns

$\tau p^2 \tau$ ↑ # left turns

p^2 ↑ # right turns.

also codes fundamental domains geodesic $[-\infty, x]$ passes through



note: quotient surface has a cusp.

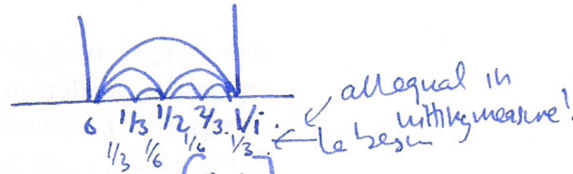
$d_{\mathbb{H}^2}(1, p^n) \approx \log(p^n)$ ← excursion into cusp.

Q: what are the continued fraction coefficients of a random number?

A1 Choose a number by doing a r.w. on trivalent tree. $\mathbb{P}(a_n = k) = \frac{1}{2^k}$.

note • largest $a_1, \dots, a_n \sim \log(n)$.

largest excursion $\sim \log(\log(n))$.

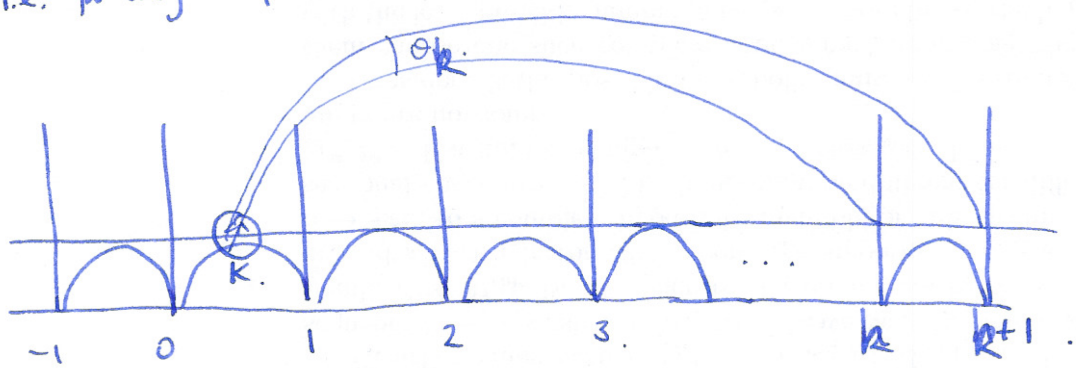


A2 Choose a number ~~by~~ ~~do~~ according to Lebesgue measure on $[0,1]$.

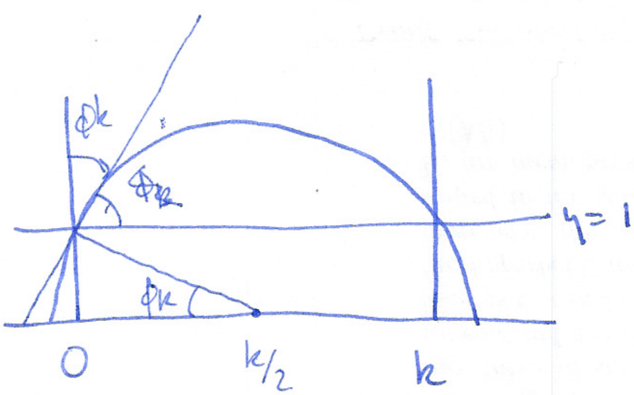
hard way: add up lots of fractions.

better way: geodesic flow on a hyp surface of finite volume is ergodic. UTS.

i.e. for any compact set $K \subset UTS$, $\gamma([0,T])$ becomes equidistributed in K.



$\mathbb{P}(a_n = k) \sim \theta_k$.



$\tan \phi_k = \frac{1}{k/2} = \frac{2}{k}$ $\phi_k \approx \frac{2}{k}$.

$\tan^{-1}(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5} \cdots$

$\theta_k \approx \phi_k - \phi_{k+1} \approx \frac{2}{k} - \frac{2}{k+1} = \frac{2}{k(k+1)} \approx \frac{2}{k^2}$.

so $\mathbb{P}(a_n = k) \sim \frac{1}{k^2}$ for Lebesgue measure

note: largest $a_1, \dots, a_n \sim n$

largest excursion $\sim \log(n)$.

Corollary Lebesgue measure + hitting measure are mutually singular.

Open questions • what is dist of a_n for π? $\sqrt[3]{2}$?

• exhibit a non-quadratic algebraic number with bounded a_n.

unbounded a_n.