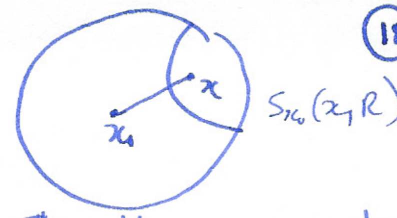
Applications

1. Exponential decay for shadows



recall: $S_{x_0}(x,R) = \{ y \in X \mid (x,y)_{x_0} \geqslant d_X(x_0,x) - R \}$

Prop^n $G \curvearrowright X$, μ non-elementary, bounded support in X, basepoint $x_0 \in X$. Then there are constants K, c s.t. $\nu(S_x(x,R)) \leq K c^{d_X(x,x)-R}$ and n^{th} $\mu_n(S_{x_0}(x,R)) \leq K c^{d_X(x,x)-R}$.

Remarks [Mathieu-Sisto][Sunderland] works for μ exponential tail.

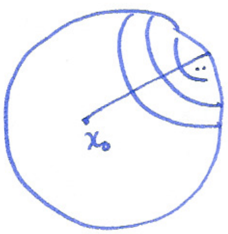
Observation ν μ-stationary is non-atomic.

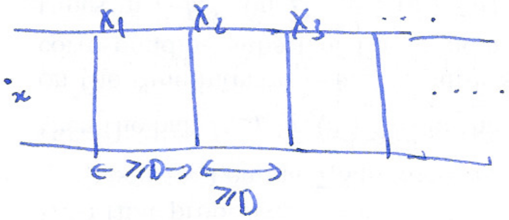
Proof: Let λ be an atom of maximal weight, then $\nu = \mu * \nu$, so $\nu(\lambda) = \sum_{g \in G} \mu(g)\nu(g^{-1}\lambda)$

$\Rightarrow$ all images of λ under $\langle \mathrm{supp}(\mu) \rangle_+$ have same weight, but there are ∞-many # $\square$.

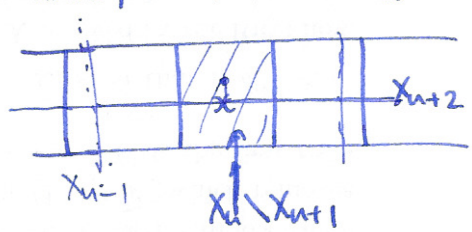
Observation $\sup_{d_X(x_0,x)-R \geqslant D} \nu(S_{x_0}(x,R)) \longrightarrow 0$ as $D \to \infty$. In particular, for any $\epsilon > 0$ there is a D s.t. $\nu(\cdot) \leq \epsilon < 1/2$.

Proof (exp. decay for shadows) Let X_n be a sequence of nested shadow sets $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ with $x_0 \notin X_1$ and $d_X(X_n, X \setminus X_{n-1}) \geqslant D$, where D is $2\max\{\mathrm{diam}_X(\mathrm{supp}(\mu)), D \text{ as above}\} + O(\delta)$





In particular, if a sample path converges into X_{n+2}, it has to hit $X_n \setminus X_{n+1}$.



Let F_n be the first hitting (improper) measure on X_n.

then $\nu(X_{n+2}) = \sum_{x \in X_n \setminus X_{n+1}} F_n(x) \nu_x(X_{n+2})$

geometric property: X_{n+2} is contained in a shadow $S_x(x_{n+2}, R)$ with $d_X(x, x_{n+2}) - R \geqslant D$.

then $\nu_x(X_{n+2}) \leq \epsilon$ for all $x \in X_n \setminus X_{n+1}$.

so $\nu(X_{n+2}) \leq \epsilon F_n(X_n)$.

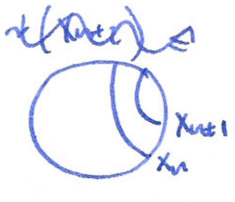
Now compare with $\nu(X_{n-1})$: $\nu(X_{n-1}) \geqslant \sum_{x \in X_n \setminus X_{n+1}} F_n(x) \nu_x(X_{n-1})$.

$X \setminus X_{n-1}$ contained in shadow $S_x(\cdot, R)$! w/ parameter $\geqslant D$.

$\geqslant (1-\epsilon) F_n(X_n)$.

so $\nu(X_{n+2}) \leq \epsilon F_n(X_n) \leq \frac{\epsilon}{1-\epsilon} \nu(X_{n-1})$, so $\nu(X_n) \leq \underbrace{\left(\frac{\epsilon}{1-\epsilon}\right)}_{\substack{c<1 \\ \text{if } \epsilon < \frac{1}{2}}}^{n/3}$ $\square$.

Final bit



$\mu_n(X_{n+1}) \geqslant (1-\epsilon)\nu(X_n)$

so $\mu_n(X_n) \leq K c^n$ $\square$.