<u>Lemma</u> $G \curvearrowright X$ metric space ν μ-stationary measure on X, $Y \subset X$.
Suppose $\exists$ sequence of +ve numbers $(\epsilon_n)_{n\in\mathbb{N}}$ s.t. for any translate fY of Y there is a sequence $(g_n)_{n\in\mathbb{N}} \in G$ s.t. the translates $\cancel{g_1}\, fY, g_1^{-1}fY, g_2 f^{-1}Y, \dots$ are all disjoint, and for each n, there is an m s.t. $\mu_m(g_n) \geqslant \epsilon_n$. Then $\nu(Y)=0$

<u>Proof</u> set $s = \sup_{f\in G} \nu(fY)$ and suppose $s>0$, set $N > \frac{2}{s}$ and $\epsilon = \min \{\epsilon_i \mid 1 \le i \le N\}$, $\epsilon_s = \frac{\epsilon}{N}$ Choose f s.t. $\nu(fY) \geqslant s - \epsilon_s$.

so $g_1^{-1}fY, g_2^{-1}fY, \dots g_N^{-1}fY$ all disjoint, and for each n $\exists$ m s.t. $\mu_m(g_n) \geqslant \epsilon_m$

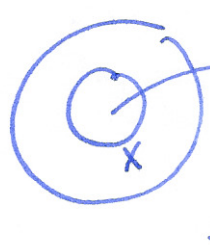
$$\nu(fY) = \sum_{h\in G} \mu_m(h)\nu(h^{-1}fY) = \sum_{h\in G} \mu_m(h)\nu(h^{-1}fY)$$

so
$$\mu_m(g_n)\nu(g_n^{-1}fY) = \nu(fY) - \sum_{h\in G\setminus g_n} \mu_m(h)\underbrace{\nu(h^{-1}fY)}_{\le s}$$
$$\geqslant s - \epsilon_s - s(1-\mu_m(g_n)).$$

$$\nu(g_n^{-1}fY) \geqslant \frac{s\mu_m(g) - \epsilon_s}{\mu_m(g)} = s - \frac{\epsilon_s}{\mu_m(g)}.$$

$g_k^{-1}fY$ all disjoint, so total measure $\geqslant N\left(s - \underbrace{\frac{\epsilon_s}{\epsilon}}_{\le \frac{s}{2}}\right) > 1$. $\#$.

<u>Corollary</u> $\nu(\overline{X}^h_F) = 0$



$\overline{B_X(x_0,R)}^h$ ← pick any lox $g \in \langle \mathrm{supp}(\mu)\rangle_+$, wlog replace g with power, with $\tau(g) \gg 2R$.

then $fY, g^{-1}fY, g^{-2}fY, \dots$ all disjoint for any f!

so $\nu(\overline{B_X(x,R)}^h) = 0$, and $\overline{X}^h_F = \bigcup_{R\in\mathbb{N}} \overline{B_X(x_0,R)}^h = 0$.

Def^{n}: Local minimum map $\phi : \overline{X}^h_\infty \to \partial X$.

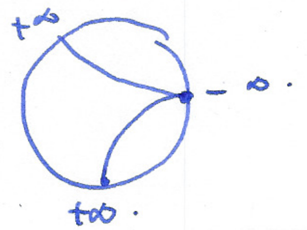
given $f \in \overline{X}^h_\infty$, there is a sequence (x_n) in X with $f(x_n) \to -\infty$.

~~In particular, this implies that $d_X(x_0, x_n) \to \infty$~~. define $\phi(f) = \overline{(x_n)} \in \partial X$.

Prop^{n}: ϕ is well defined.

Pf note · $f(x_n) \to -\infty \Rightarrow d_X(x_0, x_n) \to \infty \not\Rightarrow \overline{(x_n)} \in \partial X$!
limit may not exist!

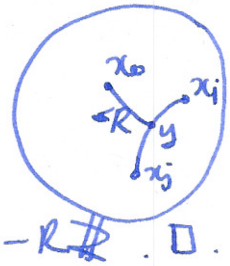
· if limit exists:



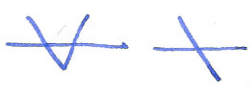
as $f|_\gamma$ is a horofunction on $\mathbb{R}$ (up to bounded error) $\exists$ exactly one point in ∂X of value $-\infty$, all others $+\infty$.

· show limit exists: want: $(x_i, x_j) \to \infty$ as $\min i,j \to \infty$.

Suppose not, then $\exists$ subsequence, R s.t. $(x_i, x_j) \leq R$.



horofunctions on $\mathbb{R}$:



have no local max.

so $f(y) \leq \max\{f(x_i), f(x_j)\} \to -\infty$, but $f(y) \geq -R$ ↯. $\square$

Fact can choose (x_n) as above to be quasigeodesic.

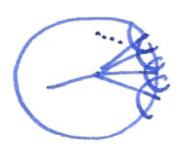
Corollary: this gives $\phi_*\nu$ μ-stationary probability measure on ∂X.

Summary $(G, \mu) \curvearrowright \overline{X}^h$

1. compactness $\Rightarrow \exists$ μ-stationary measure ν on $\overline{X}^h$
2. $\nu(\overline{X}^h_F) = 0$
3. $\int f(x)\, dw_n\nu(x)$ martingale, martingale convergence $\Rightarrow$ $w_n\nu \to$ measure ν_ω on $\overline{X}^h_\infty$.
~~4. push forward~~ use Riesz for X compact Hausdorff, $C(X)$ $(= C_b(X))$ dual is Borel measures on X.
4. push forward to $\phi_*\nu_\omega$ on ∂X.
5. claim: a.e. $(w_n x_0)$ has a subsequence (w_n'), which converges in ∂X. (*)
6. $w_n' x_0 \to \lambda \Rightarrow w_n'\nu \to \delta_\lambda$
7. ~~$w_n\nu \to \delta_\lambda$~~ w_n' convergence $\Rightarrow w_n\nu \to \delta_\lambda$.
8. $w_n\nu \to \delta_\lambda \Rightarrow w_n x_0 \to \lambda$.

Intuition for 5.: $\nu(\overline{B_X(x_0, R)}) = 0 \Rightarrow P w_n$ transient on bounded sets, in fact $\overline{B(x_0,R)}^h$.

so $\inf(Pw_n) \to -\infty$. Suppose $Pw_n \to P \in \overline{X}^h_F$:



must leave every finite set of shadows of depth $\geq -M$

$\overline{X}^h_\infty$ covered by countably many shadows
so can choose finitely many of measure $\geq 1 - \epsilon$.