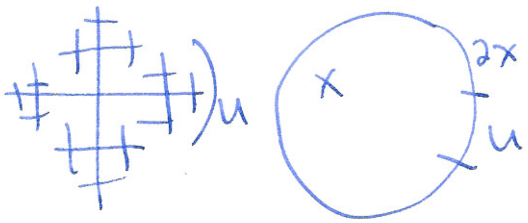
intuition: consider n, p_n, q_n to be reasonably large, and consider the next 100 choices, don't change ratio $\frac{p_n}{q_n}$ very much, so expect ratio of black to white in the next 100 to be close to p_n/q_n.

Example simple r.w. on F_2 / any random walk w/ convergence to ∂X.



define $X_n = \mathbb{P}$(r.w. starting at w_n converges into U [illegible]).

↑ sequence of r.v.s.

$X_n = \nu_{w_n}(U) = \nu(w_n^{-1}U) = w_n\nu(U)$.

claim $(X_n)_{n\in\mathbb{N}}$ is a martingale.

proof $\mathbb{E}(X_{n+1} | X_n)$ $\quad F_2$: $\nu_{w_n a}(U)\frac{1}{4} + \nu_{w_n a^{-1}}(U)\frac{1}{4} + \dots + \nu_{w_n b^{-1}}(U)\frac{1}{4}$.

$= \nu(a^{-1}w_n^{-1}U)\frac{1}{4} + \dots$

$= a\nu(w_n^{-1}U)\frac{1}{4} + a^{-1}\nu(w_n^{-1}U) + \dots = \nu_{[illegible]}(w_n^{-1}U)$

$= \nu_{w_n}(U)$ as required. by ν μ-harmonic

in general: $\mathbb{E}(X_{n+1} | X_n) = \sum_{s_{n+1}\in G} \nu_{w_n s_{n+1}}(U)\mu(s_{n+1})$.

$= \sum \nu(s_{n+1}^{-1}w_n^{-1}U)\mu(s_{n+1})$.

$= \sum s_{n+1}\nu(w_n^{-1}U)\mu(s_{n+1})$

$= \nu(w_n^{-1}U)$ by ν μ-stationary

$= \nu_{w_n}(U) = X_n$. $\square$.

Thm Convergence to boundary ∂X compact.

Let μ be a ~~nonelementary~~ prob. dist on $G \curvearrowright$ hyp, with $\langle \mathrm{supp}(\mu) \rangle_+ = G$. Then almost every sample path $(w_n)_{n\in\mathbb{N}}$ converges to a point $\partial((w_n))_{n\in\mathbb{N}} \in \partial X$.

Proof (sketch) $\overline{X} = X \cup \partial X$ compact $\Rightarrow$ $P(\overline{X})$ $\Rightarrow$ $\exists$ non-atomic μ-stationary measure ν on $\overline{X}$.

consider sequence of measures $w_n\nu$ on ∂X. pick $f \in C(\overline{X})$, then

$X_f = \int_{\overline{X}} f(x)\, d(g_n\nu)(x)$ is a martingale, and converges for a.e. $\omega \in \Omega$.

$C(\overline{X})$ separable $\Rightarrow$ $P(\overline{X})$ separable, i.e. has countable dense set $\{f_i\}_{i\in\mathbb{N}}$.

so for a full measure set of Ω ~~X_{f_i}~~ $X_{f_i,n}$ converges for all f_i $\Rightarrow$

$w_n\nu$ converges to a measure $\nu(\omega) \in P(\overline{X})$. [Riesz-rep thm].

(positive linear functionals on $C(\overline{X}) \longleftrightarrow$ Borel measures on $\overline{X}$.)

$\overline{X}$ compact $\Rightarrow$ every $(w_n x_0)$ has a convergent subsequence, r.w. transient $\Rightarrow$ must convergence to $\lambda \in \partial X$. **Claim**: if $w_n \to \lambda \in \partial X$ then $w_n\nu \to \delta_\lambda \in P(\partial X)$

Proof: $w_n \to \lambda$ has subsequence s.t. $w_n\eta \to \lambda$ for all but one point $\eta \in \partial X$.

spse two points $\eta, \eta' \not\to \lambda$, then avoid open set $U \ni \lambda$.

$\Rightarrow w_n \notin U$ for all n ⨳. so for all $\lambda \in V \not\ni \eta$ $\quad w_n\nu(V) =$

$\nu(w_n^{-1}V)$ converges to $\nu(\partial X \setminus \eta) = 1 \Rightarrow w_\omega\nu = \delta_\lambda$. $\square$.

get convergence of sequence from convergent sub sequence.

claim if $w_n\nu \to \delta_\lambda$ then $w_n \to \lambda$.

spse not, two convergent subsequences $w_{n_1}, w_{n_2} \to \lambda_1 \neq \lambda_2 \Rightarrow w_{n_i}\nu \to \delta_{\lambda_i}$, $\lambda_i \neq \lambda_j$

⨳. $\square$.

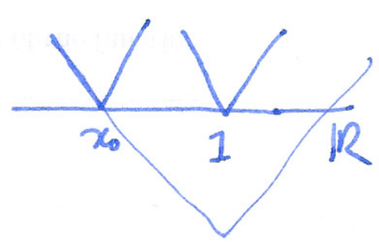
Aim fix this to work for $G \curvearrowright X$ hyp, not locally compact.

Step 1 $X \cup \partial X$ compact $\Rightarrow P(X \cup \partial X)$ compact $\Rightarrow \exists$ ν-stationary measure.

problem $X \cup \partial X$ not compact.

solution: use horofunction boundary $\overline{X}^h$ always compact!

Defn horofunction at $x \in X$ is $\rho_x(\cdot) = d_X(\cdot, x) - d_X(x_0, x)$



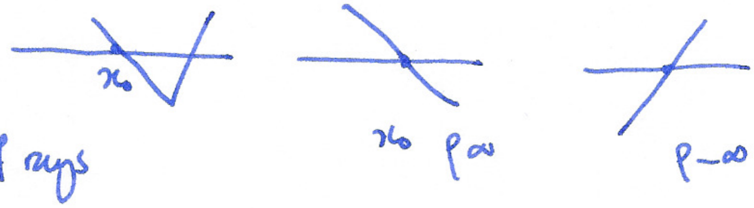
$\rho : X \to C(X) \leftarrow$ 1-Lipschitz functions w/ $\rho(x) = 0$.

Def$^{\underline{n}}$ horofunction compactification $\overline{X}^h = \overline{\rho(X)}$ in $C(X)$

$\uparrow$ topology of uniform convergence on <u>compact</u> sets (same as pointwise convergence)

Observation $G \curvearrowright \overline{X}^h$ by homeos.

Example $\overline{\mathbb{R}}^h = \mathbb{R} \cup \{\pm\infty\}$



Example X = countable wedge of rays

Gromov $\partial X = \mathbb{N}$

$\overline{X}^h \setminus X = \mathbb{N}$! <u>note</u> $\rho_{x_n} \to \rho_{x_0}$ if x_n leaves every finite set of rays.

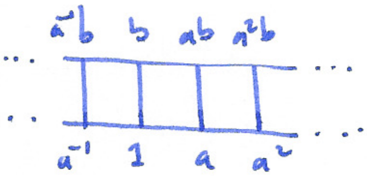
Prop$^{\underline{n}}$ X Gromov hyp, γ geodesic (quasi), $\rho|_\gamma$ = horofunction on $\mathbb{R}$, up to bounded error and vertical translation.

Th$^{\underline{m}}$ G countable $\curvearrowright$ X hyperbolic, $\langle \text{supp}(\mu) \rangle_+$ non-elementary, then $(w_n x)$
$\uparrow$ isometries
converges to a point in ∂X a.s.

Proof $\overline{X}^h$ compact ! so $\exists$ μ-stationary measure ν on $\overline{X}^h$ ....

warning : · sample paths do not converge in $\overline{X}^h$
· $\overline{X}^h$ <u>not</u> QI-invariant

Example $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
$a \quad b$



$\rho_{a^n b}(b) = \underbrace{d(a^n b, b)}_{n} - \underbrace{d(a^n b, 1)}_{n+1} = -1$

$\rho_{a^n}(b) = \underbrace{d(a^n, b)}_{n+1} - \underbrace{d(a^n, 1)}_{n} = +1$

$F_2 \oplus \mathbb{Z}/2\mathbb{Z} \curvearrowleft \overline{X}^h$ = two copies of ∂F_2.

Proof (continued). want $\overline{X}^h \longrightarrow X \cup \partial X$.
do probability $\quad$ $\uparrow$ do geometry

can decompose $\overline{X}^h = \overline{X}^h_F \sqcup \overline{X}^h_\infty$
$\uparrow$ $\inf(\rho) > -\infty$ $\quad$ $\uparrow$ $\inf(\rho) = -\infty$.

useful facts <u>Prop$^{\underline{n}}$</u> ① $\nu(\overline{X}^h_F) = 0$

② $\exists\, \phi : \overline{X}^h_\infty \longrightarrow \partial X$.
$\nu \longmapsto \phi_*\nu$.

Proof ① <u>Prop$^{\underline{n}}$</u> Let $(G,\mu) \curvearrowright (X,\nu)$ ν μ-stationary, and let $Y \subseteq X$ s.t. for any translate gY, there is, $\varepsilon > 0$ and an element $h(g,Y) \in G$ s.t. $hgY \cap gY = \phi$ [illegible]