Martingales X_n sequence of random variables w/ values in $\mathbb{R}$ (not nec indep, ident dist etc).

Example X_n = location of nearest neighbour r.w. on $\mathbb{Z}$ at time n. ($X_n = Y_1 + \dots + Y_n$)
indep. iid $Y_i = \pm 1$ w/ prob $1/2$.

Defn $(X_n)_{n \in \mathbb{N}}$ is a martingale if $\mathbb{E}(X_{n+1} | X_n) = X_n$

expectation of conditional prob. also a random variable! random variable.

Example: $X_{n+1} = \begin{cases} X_n + 1 & \text{w/ prob } 1/2 \\ X_n - 1 & \text{w/ prob } 1/2 \end{cases}$

so $\mathbb{E}(X_{n+1} | X_n) = (X_n + 1)\frac{1}{2} + (X_n - 1)\frac{1}{2} = X_n$, as required □.

Non-example: nearest nbr r.w. on F_2, set $X_n = d_{Cay(F_2)}(1, w_n)$

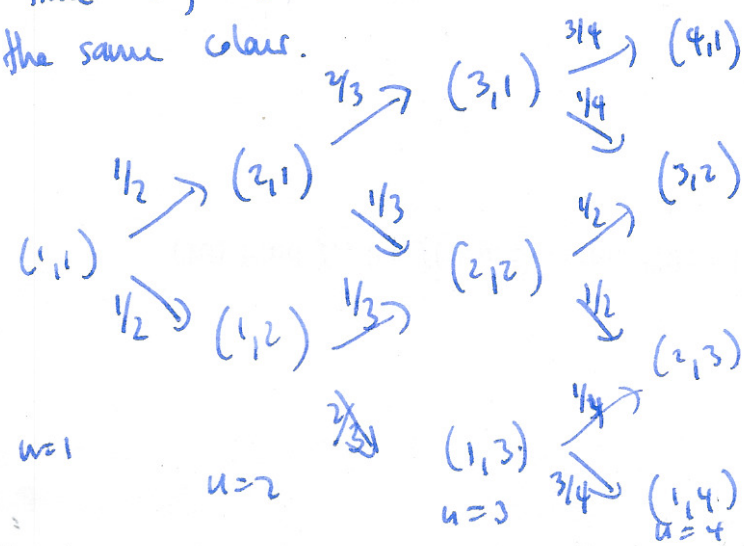
$\mathbb{E}(X_{n+1} | X_n) \approx \frac{3}{4}(X_n + 1) - \frac{1}{4}(X_n - 1) = X_n + \frac{1}{2}$ so not martingale!
also $\frac{1}{n} X_n$ for n.r.w on $\mathbb{Z}$ not martingale.

Remark submartingale: $\mathbb{E}(X_{n+1} | X_n) \geq X_n$
supermartingale: $\mathbb{E}(X_{n+1} | X_n) \leq X_n$ } won't need these.

Thm (martingale convergence) Let $(X_n)_{n=1}^{\infty}$ be a bounded martingale, i.e. $\exists\, B$ s.t. for all n $|X_n| \leq B$. Then $(X_n)_{n=1}^{\infty}$ converges almost surely.

Notation: $X_n : \Omega \to \mathbb{R}$, so really family of sequences $(X_n(\omega))_{n \in \mathbb{N}}$ so $(X_n(\omega))_{n \in \mathbb{N}}$ converges almost surely.

Example (Polya's urn) at time $n = 1$ contains 1 black, 1 white ball.
at time n, remove 1 ball chosen uniformly at random, and replace it with one of the same colour.



Exercise dist of (p,q) at time n is uniform $\frac{1}{n}$ for each (p,q), $p+q = n+1$

Martingale converge theorem implies almost every sequence $\frac{p_n}{q_n}(\omega)$ converges to a real number $r(\omega) \in [0,1]$ (in fact uniformly dist).