Q: given two groups G, H is G isomorphic to H? $G \cong H \Rightarrow Cay(G) \underset{QI}{\cong} Cay(H)$.

Example. $F_2 \not\cong \pi_1(\text{closed surface}) \not\cong \mathbb{Z}$.

Q: do G, H have isomorphic finite index subgps? $G' \cong H' \Rightarrow Cay(G) \underset{QI}{\cong} Cay(H)$.
$G' < G$ $H' < H$. (Def^n commensurable)

Example all closed surface groups QI; all F_r QI.

~~there are~~ all closed hyp. M^3 groups QI, but there are ones which are not commensurable. (consider trace field / alg. number field of subgp of $PSL(2,\mathbb{C})$).

Example (Margulis super-rigidity, simplest possible case) rules out homomorphism $SL_n\mathbb{Z} \to SL_m\mathbb{Z}$ for $n > m$.

Thm Λ lattice in $SL_n\mathbb{Z}$, and $\phi : \Lambda \to SL_m\mathbb{Z}$ homomorphism.
Then this extends to a Lie group homomorphism $\Phi : SL_n\mathbb{R} \to SL_m\mathbb{R}$.

Thm [Farb–Kaimanovich–Masur] any homomorphism from Λ higher rank lattice $\to$ MCG(surface) has finite image.

Thm [Haettel] Any action of a higher rank lattice on a hyperbolic space is elementary.

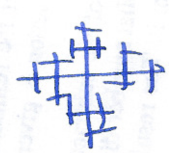
Remark ($\Rightarrow$ [FKM]) (uses r.w.s!) sketch. $\Lambda \xrightarrow{\phi} MCG(S) \curvearrowright \mathcal{C}(S)$.

r.w. ϕ(r.w) $\leftarrow$ r.w!

$\exists \mu$ s.t. r.w makes sublinear progress

non-elementary, every μ makes linear progress #.

Recall nearest neighbour r.w on F_2: transience $\Rightarrow$ convergence to boundary



Thm [Kaimanovich? [Furstenberg–Margulis]] G Gromov hyperbolic group, μ prob dist on G s.t. $\langle supp(\mu) \rangle_+ = G$, then almost every sample path $(w_n)_{n\in\mathbb{N}}$ converges to ∂G.

Thm [M–Tiozzo] G countable group acting by isometries on X Gromov hyperbolic.
μ prob dist on G s.t. $\langle supp(\mu) \rangle_+$ non-elementary. Then almost every sample path $(w_n)_{n\in\mathbb{N}}$ converges to a point in ∂X.

Applications: linear progress, $\frac{1}{n} d(x_0, w_n x_0) \to \ell > 0$ a.s., $\mathbb{P}(w_n \text{ loxodromic}) \to 1$ as $n \to \infty$.
[Haettel] result.

Reference: [Aliprantis + Border].

Simplification: assume $X, \partial X$ separable, i.e. contain countable dense subset. (not necessary [Sisto]).

Fact $X \cup \partial X$ compact $\Leftrightarrow$ $P(X \cup \partial X)$ is compact, in weak-* topology.

weak-* topology: Let $C(X)$ be the space of continuous functions on X. Then $\mu_n \to \mu$ iff $\int f\, d\mu_n \to \int f\, d\mu$ for all continuous functions $f \in C(X)$.

Fact: $X \cup \partial X$ separable $\Leftrightarrow$ $C(X \cup \partial X)$ separable, i.e. countable dense subset, so suffices to check countably many $f_i \in C(X)$. **Fact** X compact $C(X) = C_b(X) \Leftarrow$ use this if X not compact.

Let $G \curvearrowright (X, \nu)$ probability space.

Warning action is measure preserving if $\nu(U) = \nu(gU)$ for all open sets U. !!
we will never think about this!

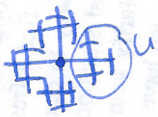
Defn $(G,\mu) \curvearrowright (X,\nu)$ ν is μ-stationary if $\nu(U) = \sum_{g \in G} \mu(g)\, \nu(g^{-1}U)$.

Defn convolution $\mu * \nu = \sum_{g \in G} \mu(g)\, g\nu$, where ~~$g\nu(U)$~~ $g\nu$ is the push forward of ν.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ (X,\nu) & & (X, g\nu) \end{array}$$

so ν is μ-stationary if $\nu = \mu * \nu$.

$\nu(g^{-1}U) = g\nu(U)$.

Example hitting measure on ∂F_2. 

$$\nu(U) = \sum_{g \in G} \mu(g)\, \nu(g^{-1}U)$$
$$= \tfrac{1}{4}\left(\nu(aU) + \nu(a^{-1}U) + \nu(bU) + \nu(b^{-1}U)\right)$$

= prob. of converging to U from next step of the random walk.

Propn $X \cup \partial X$ compact $\Rightarrow \exists$ μ-stationary measure.

Proof (Cesaro averages) Let $\nu_n = \frac{1}{n}(\mu + \mu_2 + \mu_3 + \cdots + \mu_n)$.

then $\cancel{\mu * \nu\ \nu_{n+1}}$ $\mu * \cancel{\mu_n} \nu_n = \frac{1}{n}(\mu_2 + \mu_3 + \cdots \mu_{n+1})$.

so $\| \nu_n - \mu * \nu_n \| = \| \frac{1}{n}(\mu - \mu_{n+1}) \| \leq \frac{2}{n} \to 0$ as $n \to \infty$.

↑ absolute difference norm on prob. measures. □.

setup. $(G,\mu) \curvearrowright (X,\nu)$ ν μ-stationary

key fact: consider sequence $(w_n) = (g_1 \cdots g_n)$ in G. Then ~~$X_n = \int f_i$~~ gives sequence of measures $w_n\nu = (g_1 \cdots g_n)\nu$ in $P(X \cup \partial X)$. Then $X_n = \int f(x)\, dw_n\nu$ is a **martingale** and converges almost surely by the martingale convergence thm.

$\Rightarrow$ X_n converges for a.e. sample path, so apply to countable dense set of f_i in $C(X \cup \partial X)$

$\Rightarrow$ $w_n \nu \to \lambda(w_n)$ measure in $P(X \cup \partial X)$ a.s.