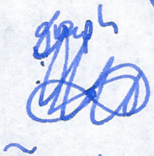
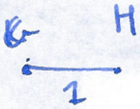
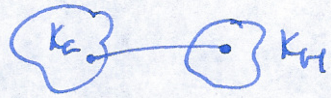
Background graphs of groups

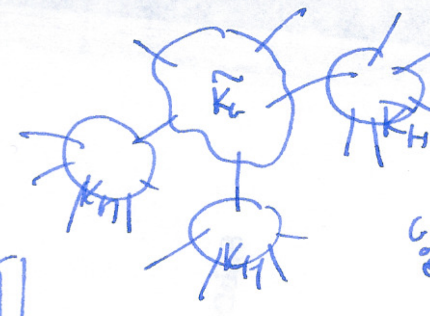
examples • **free products** $G * H$ topological model: $K(G,1) \vee K(H,1)$.



Fact given a group G, there is a topological space X $K(G,1)$ with $\pi_1 X = G$ and $\tilde{X}$ contractible.

graph


X:


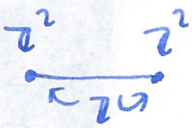
$\tilde{X}$


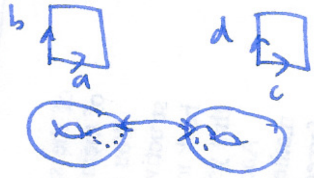
• **amalgamated products**.

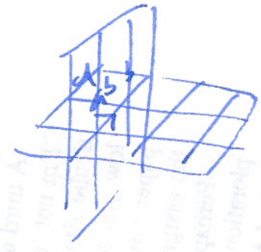
example $\mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^2$.

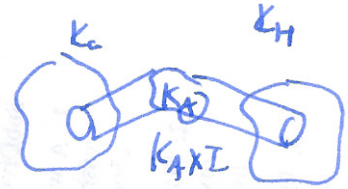
$\langle a,b \rangle * \langle c,d \rangle$

$b = c$.

graph


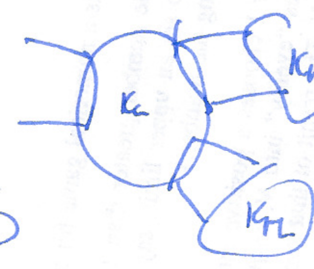


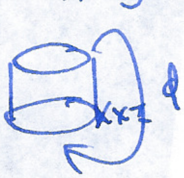


X


• **mapping tori/HNN extension**

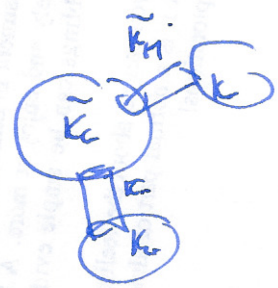
$G *_A H$ ↺





φ.

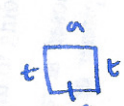
$G *_H$ ↺

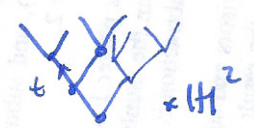


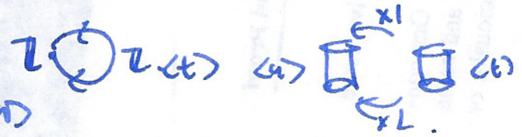
Fact: a graph of groups acts on a tree.

Example $PSL_2\mathbb{Z} \cong \mathbb{Z}_2 * \mathbb{Z}_3$. **Fact** $SL_3\mathbb{Z}$ does not act on a tree.

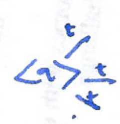
Example $BS(1,2) = \langle a,t \mid t^{-1}at = a^2 \rangle$. Cayley graph:







tree:



∂X = Cantor set $\cup$ $\{pt\}$.

parabolic action: all loxodromics preserve $\{pt\}$.

Thm [Bass-Serre] Any action of a higher rank lattice on a tree has a fixed pt.

Defn $G \curvearrowright X$ is WPD if $g \in G$ is WPD.

Defn $g \curvearrowright X$ is WPD if for all $K > 0$, for all $x \in X$ $\exists\, M \in \mathbb{N}$ s.t.

$$\left|\{g \in G \mid d(x, gx) \le K,\ d(h^M x, g h^M x) \le K\}\right| < \infty$$

$$\mathrm{stab}_K(x) \cap \mathrm{stab}_K(h^M x).$$

Thm [Osin]. $G \curvearrowright X$ acylindrically $\cancel{\Leftrightarrow} \Rightarrow G \curvearrowright X$ WPD.
$G \curvearrowright X$ WPD $\Rightarrow G \curvearrowright X'$ acylindrically.

Thm [Osin] if $G \curvearrowright X$ acylindrically then no parabolics, and furthermore, exactly one of the following occurs:
1) G has bounded orbits
2) G is virtually cyclic and contains a loxodromic
3) G non-elementary

Example Defn $MCG(S) = \mathrm{Diff}_+(S)/\text{isotopy}$. S orientable surface (finite type).

Examples: Dehn twists  Involution 

periodic maps  order 4  order 6

Fact $MCG(T^2) = SL(2, \mathbb{Z})$ (action on homology).

Defn complex of curves: $\mathcal{C}(S)$ simplicial complex: vertices: isotopy classes of scc.
simplices ~~edges~~: ~~disjoint~~ spanned by disjoint curves.

Example $\mathcal{C}(T^2)$ = Farey graph. (edges spanned by minimally intersecting curves).

Thm [Masur-Minsky] $\mathcal{C}(S)$ is Gromov hyperbolic. pA $\leftrightarrow$ loxodromics.

Thm [Bowditch]... $G \curvearrowright \mathcal{C}(S)$ is acylindrical.

variations: complex of separating / non-separating curves, pants complex....

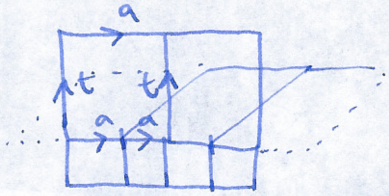
Fact $MCG(S)$ is not relatively hyp. **Q**: what is curved Cayley graph? not δ-hyp.

$$\begin{array}{llll} G \curvearrowright & \text{Teich} & & \\ & \downarrow & & \\ & \text{Teich}, d_{WP} \cong \text{QI pants complex [Brock]} & & CAT(0) \text{ (incomplete)} \\ & \downarrow & & \\ & \mathcal{C}(S). & & \text{Gromov hyp.} \end{array}$$

Notes on $BS(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle = G$ $ab(BS(1,2)) \cong \mathbb{Z} = \langle t \rangle$.

Cayley graph:

a, t preserve direction of horizontal/vertical lines $\Rightarrow$ so does G.



let $\phi : G \twoheadrightarrow \mathbb{Z}$.

$\leftarrow$ Gromov boundary

note: not Gromov hyperbolic, e.g.

side view:

Fact: Bass-Serre tree.

unstable geodesics / thin triangles.

local picture:

$$\begin{array}{ccc} t\langle a\rangle & & at\langle a\rangle \\ \nwarrow & & \nearrow \\ & \langle a\rangle & \\ & \uparrow & \\ & t^{-1}\langle a\rangle & \end{array}$$

normal form:

$t^k \underbrace{\cancel{(at)^\ell}}_{w} \langle a\rangle$.

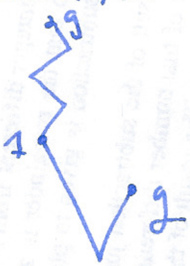
use: $at^{-1} = t^{-1}a^2$ shuffle all t^{-1} to the front.

$a^2t = ta$.

or some word in t, at $w\langle a\rangle$.

claim $G \curvearrowright T$ isometry: check local picture under g.

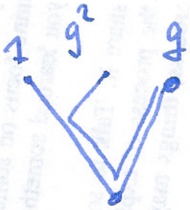
think about geodesics in binary tree:



at most 1 local min.

dynamics of $G \curvearrowright T$:

suppose $\phi(g) = 0$, consider:



$\Rightarrow$ local min fixed

$\Rightarrow$ g finite order

$\phi(g) \neq 0$ wlog $\phi(g^n) \to +\infty$

$\phi(g^{-n}) \to -\infty$. $\leftarrow$ any two descending paths coincide

$\Rightarrow$ all have common fixed pt in ∂T.

$\frac{1}{n}\phi(g^n)$ lower bound on $d(1, g^n) \Rightarrow \tau(g) > 0 \Rightarrow$ loxodromic.

u_{pp}

stabilizes.