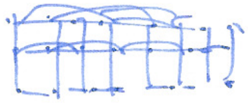
Def^n G is <u>Gromov hyp</u> if S finite generating set and $\mathrm{Cay}(G,S)$ is δ-hyp.

Def^n G is <u>relatively hyp</u> if $\{H_i\}$ collection of subgroups and $\mathrm{Cay}(G,S) \cup$ cusps is Gromov hyperbolic:



$\mathrm{cusp}(H,S) =$ graph: vertices: $H \times \mathbb{N}$.

edges: $((h,k),(hs,k))$ for all h,s,k. (vertical).

$((h,k),(h',k))$ if $d(h,h') \leq 2^k$ (horizontal)

Thm [....] this is equivalent to all the ~~the~~ Def^ns of Gromov hyp.

~~Def^n G ⤳ X acylindrical~~

Def^n G is <u>weakly hyperbolic</u> if $G \curvearrowright X$, Gromov hyp by isometries

usually want <u>non-elementary</u>: contains two <u>loxodromic</u> element

$\langle a,t \mid t^{-1}at = a^2 \rangle$
$\parallel$
e.g. $BS(1,2)$
$\curvearrowright$ Bass-Serre tree.

<u>Classification of isometries</u> 3

periodic: bounded orbits

parabolic: $\tau(g) = 0$

loxodromic / hyperbolic: $\tau(g) > 0$

$$\tau(g) = \lim_{n\to\infty} \frac{1}{n} d_X(x, g^n x)$$

<u>Classification of group actions</u> [Gromov] / [Osin]

$G \curvearrowright X$, hyp let $\Lambda(G) =$ limit set of G, i.e. $\overline{\{Gx_0\}} = \overline{\{gx_0 \mid g \in G\}}$.

1) $|\Lambda(G)| = 0 \iff G$ has bounded orbits called <u>elliptic action</u>.

2) $|\Lambda(G)| = 1 \iff G$ has unbounded orbits, contains no loxodromics <u>parabolic action</u>.

3) $|\Lambda(G)| = 2 \iff G$ contains a loxodromic element, all loxodromics have same limit point.

4) $|\Lambda(G)| = \infty$ a) any two ~~limit points~~ loxodromics of G have a common limit point in ∂X, called <u>quasi-parabolic</u>

b) G contains ~~at 2~~ ∞-many independent loxodromics } non-elementary

$\iff$ at least two.

4

Def^n $\mathrm{stab}_K(x) = \{g \in G \mid d(gx,x) \leq K\}$.

Def^n $G \curvearrowright X$ <u>acylindrical</u> if for all K, $\exists\, N(K), R(K)$ s.t. for all x,y with $d_X(x,y) \geq R(K)$, $|\mathrm{stab}_K(x) \cap \mathrm{stab}_K(y)| \leq N(K)$

classification of isometries

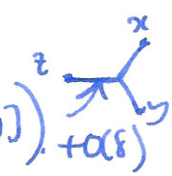
$G \curvearrowright X$ hyp.
group ↑ isometries.

Defn g is elliptic if g has bounded orbits ($\Rightarrow \tau(g)=0$)

Defn g is parabolic if g has unbounded orbits and $\tau(g)=0$

Defn g is loxodromic (hyperbolic) if $\tau(g)>0$.

useful properties **Propn** g loxodromic has exactly two fixed points in ∂X and acts as a translation along the quasi-axis connecting them.



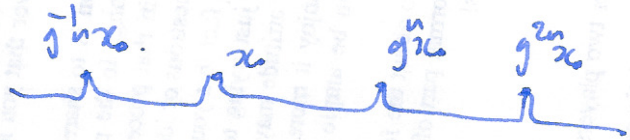
recall: · Gromov product $(x,y)_z = \frac{1}{2}\left(d(x,z)+d(y,z)-d(x,y)\right) = d(z,[x,y]) + O(\delta)$

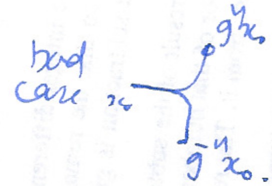
· given $K \geq 0$ $\exists$ L, Q, c s.t. a sequence $(x_i)_{i\in\mathbb{Z}}$ w/ $d(x_i,x_{i+1}) \geq L$ and $(x_{i-1},x_{i+1})_{x_i} \leq K$ is (Q,c)-quasigeodesic ($L = 2K+O(\delta)$).

Proof (of Propn) $\tau = \tau(g) = \lim_{n\to\infty} \frac{1}{n} d(x, g^n x) > 0$ so for all $\epsilon > 0$ there is an N

s.t. $(\tau-\epsilon)n \leq d(x, g^n x) \leq (\tau+\epsilon)n$ for all $n \geq N$.

consider $(g^n x_0)_{n\in\mathbb{Z}}$.





check · $d(x, g^n x) \geq (\tau-\epsilon)n$.

$$\cdot\ (g^{-n}x_0, g^n x_0)_{x_0} = \frac{1}{2}\left(d(x_0, g^{-n}x_0) + d(x_0, g^n x) - d(g^{-n}x_0, g^n x)\right).$$

$$\leq \frac{1}{2}\left((\tau+\epsilon)n + (\tau+\epsilon)n - 2(\tau-\epsilon)n\right).$$

$$\leq 2\epsilon n.$$

($d(g^{-n}x_0, g^n x) = d(x, g^{2n}x)$.)

for ϵ sufficiently small $\underbrace{(\tau-\epsilon)n}_{L} >> \underbrace{4\epsilon n}_{2K} + O(\delta)$ as required. □

$(g^n x)_{n\in\mathbb{Z}}$ quasi-geodesic, g-invariant.
dynamics determined by nearest point projections.