Def^n A <u>quasi geodesic</u> (<u>ray</u>) is a quasi-isometric embedding $[a,b]$ $\mathbb{R} \to (X,d)$
$\mathbb{R}_{\geq 0} \to (X,d)$.

<u>Examples</u> $\mathbb{R} \hookrightarrow \mathbb{R}^2$, any geodesic in $\mathbb{H}^2$

exp. spiral in $\mathbb{R}^2$. (yuck!).

<u>Fact</u> (Morse Lemma) (geodesic stability) if $\gamma : [a,b] \to (X,d)$ is a (K,c) quasigeodesic, then $\exists\, L(K,c)$ s.t. $\gamma \subset N_L([a,b])$ a geodesic from a to b in (X,d). if X Gromov hyp'.

(not true in $\mathbb{R}^2$!).

Def^n (X,d) Gromov hyp geodesic locally compact. Gromov boundary ∂X = equivalence classes of geodesic rays: γ, γ' equivalent if $\gamma \subset N_K(\gamma')$.

Def^n (X,d) Gromov hyp. ∂X = equivalence classes of quasi geodesic rays.

Def^n (X,d) Gromov hyp. a sequence (x_i) is <u>convergent</u> if $(x_i, x_j)_{x_0} \to \infty$ as min $i,j \to \infty$

$\partial X = \{\text{convergent sequences}\}/\sim \quad (x_i) \sim (y_i)$ if $(x_i, y_i)_{x_0} \to \infty$ as $i \to \infty$.

<u>Fact</u> all these definitions agree when they overlap.

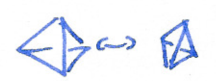
<u>Examples</u>. $\mathbb{R}$. $\partial \mathbb{R} = \{\pm\infty\}$ $\quad \mathbb{H}^n$, $\partial \mathbb{H}^n$ = sphere at infinity $\quad$ T trees, ∂T = space of ends.

<u>Facts</u>: there is a topology on the boundary, ∂X is metrizable.

<u>Problem</u> lots of groups aren't hyperbolic.
<u>Solution</u> look at groups that act on Gromov hyp spaces. <u>Example</u> surface vs. punctured surface

<u>Example</u> fig-8 knot complement.

$\leftarrow \pi_1(S^3 \setminus K) = \langle a,b \mid a^{-1}bab^{-1}ab = ba^{-1}ba \rangle$ =



contains $\mathbb{Z}^2$ subgp $\Rightarrow$ not Gromov hyp.

but ~~$\pi_1(S^3$~~ $G \curvearrowright \mathbb{H}^3$ discretely $\mathbb{H}^3/G$ not ~~finite~~ compact (finite volume).

<u>note</u> orbit map $G \to \mathbb{H}^3$, $g \mapsto gp$ gives induced metric on G which is hyperbolic but not geodesic

Q.I. embedded in G but not in $\mathbb{H}^2$.

(lots of examples like this: all hyp knots, all cusped hyp. manifolds).

<u>Example</u> $MCG(S) = \mathrm{Diffeo}_+(S)/\text{isotopy}$.

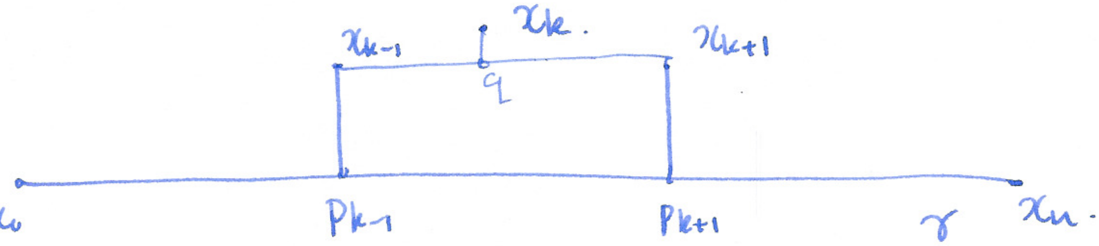
$\gamma_i = [x_{i-1}, x_i]$ ①

Lemma given K, $\exists\, L, Q, c$ s.t. if forall i $(x_i, x_{i+2})_{x_{i+1}} \leq K$

and forall i $d_X(x_i, x_{i+1}) \geq L$ $(\geq 2K + O(\delta))$

then $\{x_i\}$ are a (Q, c)-quasigeodesic.

Proof let γ be the geodesic from x_0 to x_n



claim all x_i contained in bounded nbd of γ

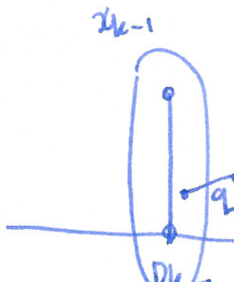
let x_k be furthest point from γ, consider x_{k-1}, x_{k+1}

with nearest point projections p_{k-1}, p_{k+1}

let q be nearest point on $[x_{k-1}, x_{k+1}]$ to x_k.

then $d_X(x_k, q) \approx (x_i, x_{i+1})_{x_i} \leq K + O(\delta)$.

by thin triangles q lies in a 2δ-nbd of $[x_{k-1}, p_{k-1}] \cup [p_{k-1}, p_{k+1}] \cup [p_{k+1}, x_{k+1}]$



as $d_X(x_{k-1}, x_k) \geq L \geq 2K + O(\delta)$

q can't be close to x_{k-1}.

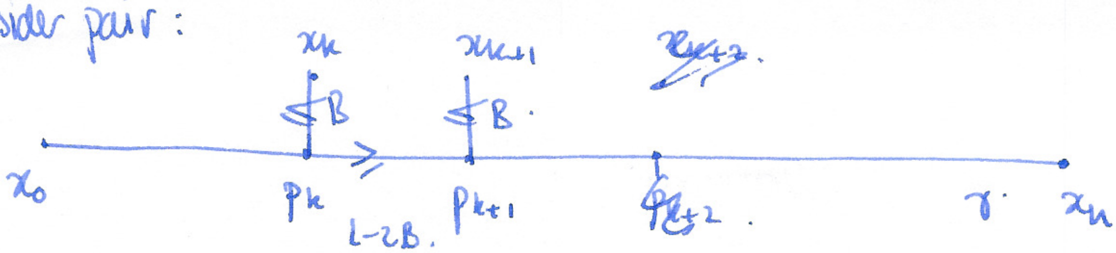
but then x_k is closer to γ than x_{k-1} ~~#~~

Claim.

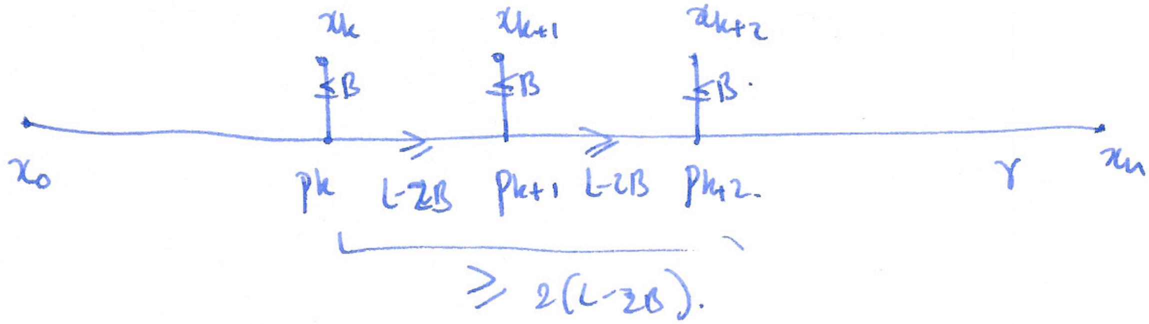
so q lies within 2δ of γ $\Rightarrow$ $d_X(x_k, \gamma) \leq K + O(\delta) + 2\delta$.

B. $\square$ claim.

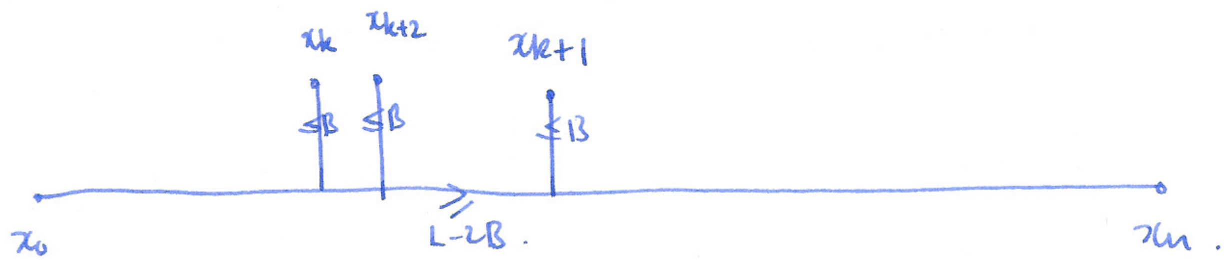
consider pair:



consider triple: right order:



wrong order:



but then $(x_k, x_{k+2})_{x_{k+1}} \geq L-2B \geq B \geq K$ #

(if $L \geq 3B \geq 3(K+O(\delta)+2\delta)$)

□.

Corollary if g is an isometry, then

$$\tau(g) \simeq d(x_0, gx) - 2(g^{-1}x, gx)_{x_0} + O(\delta).$$