Probability background

intuition: toss a coin, heads or tails equally likely

mathematical model: set $\{H,T\}$, measure $\mathbb{P}$: subsets $\longrightarrow \mathbb{R}$.

$\phi \longmapsto 0$
$\{H\} \longmapsto 1/2$
$\{T\} \longmapsto 1/2$
$\{H,T\} \longmapsto 1$.

want: if $A \cap B = \phi$ then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

$\mathbb{P}(\text{total set}) = 1$.

This works for finite / countable sets:

U countable, a probability measure is just a function $\mathbb{P} : U \to \mathbb{R}$ w/ $\mathbb{P}(U) = 1$.

with if $A \subseteq U$ $\mathbb{P}(A) = \sum_{u \in A} \mathbb{P}(u)$.

Example toss coin 3 times, independently, just get product $(\{H,T\}, \mathbb{P})^3$.

$\mathbb{P}^3(A_1 \times A_2 \times A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$.

HHH 1/8
HHT 1/8
HTH
⋮
TTT 1/8.

$\mathbb{P}(\text{get two heads}) = \mathbb{P}(HHT, HTH, THH) = 3/8$.

Problem: ~~toss co~~ (infinite) sequence of coin tosses: space is $\{0,1\}^{\mathbb{N}}$ (or $\{0,1\}^{\mathbb{Z}}$)

not countable (cardinality of $[0,1]$) ← too many subsets to assign measures to each one.

Solution: only assign measures to special subsets, called measurable sets.

Example $(\{0,1\}, \mathbb{P})^{\mathbb{N}}$ measurable sets are: $\cdots \{0,1\} \times \{0,1\} \times A \times \{0,1\} \times \{0,1\} \times \cdots$

+ complements, countable unions and intersections.] ← this is called a σ-algebra.

Intuition: if you can describe your set w/out using axiom of choice, it's measurable.

Q: what is a random variable?

A: Defn a random variable is a function on a probability space.

Q: when are two random variables independent? ~~if~~ $(U_1, \mathbb{P}_1) \xrightarrow{X_1} \mathbb{R}$ $(U_2, \mathbb{P}_2) \xrightarrow{X_2} \mathbb{R}$.

A: doesn't make sense unless the random variables are defined on the same probability space. $[\mathbb{P}(X_1 \in A, X_2 \in B) = \mathbb{P}(X_1 \in A)\mathbb{P}(X_2 \in B)]$.

Example model for independent coin tosses is $(\{0,1\}, \mathbb{P})^{\mathbb{N}}$. Each coin toss is

$X_i : (\{0,1\}, \mathbb{P})^{\mathbb{N}} \longrightarrow$ projection to i-th factor. if ~~X_i~~ $i \neq j$, X_i, X_j independent by construction.

⑤

<u>Expectation</u> (for r.v. taking values in $\mathbb{Z}$, $\mathbb{R}$). $X : (\mathcal{U}, \mathbb{P}) \to \mathbb{Z}$ $\mathbb{E}(X) = \sum_{n \in \mathbb{Z}} n \mathbb{P}(X = n)$

$X : (\mathcal{U}, \mathbb{P}) \to \mathbb{R}$ <u>assume</u> this map is <u>measurable</u> and takes σ-algebra of $\mathcal{U}$ into standard σ-algebra of $\mathbb{R}$ generated by open sets. then $\mathbb{E}(X) = \int_{\mathbb{R}} x \, d\mathbb{P}_X(x)$. $\mathbb{P}_X$ = push forward of measure on X to $\mathbb{R}$, ie $\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A))$.

<u>Setup</u> G <u>countable</u> group μ probability dist on G. ($\mu : G \to \mathbb{R}$ w $\mu(G) = 1 = \sum_{g \in G} \mu(g)$).

step space : sequence of indep μ-dist r.v.s. $(G, \mu)^{\mathbb{N}}$ or $(G, \mu)^{\mathbb{Z}}$.

$(g_1, g_2, g_3, \dots) \in (G, \mu)^{\mathbb{N}} = (G^{\mathbb{N}}, \mu^{\mathbb{N}})$. (in simple r.w. on $\mathbb{Z}$ $(+1, +1, -1, +1, \dots)$.)

path space : $w_n = g_1 g_2 g_3 \cdots g_n$.
$w_0 = 1$.
$w_{-n} = g_0^{-1} g_{-1}^{-1} \cdots g_{-n+1}^{-1}$

as a set : $G^{\mathbb{N}}$ or $G^{\mathbb{Z}}$.

$(G, \mu)^{\mathbb{N}} \xrightarrow{\phi} (G^{\mathbb{N}}, \mathbb{P})$

$(g_1, g_2, \dots) \mapsto (1, g_1, g_1 g_2, \dots, w_n, \dots)$

$\mathbb{P}$ = push forward of $\mu^{\mathbb{N}}$.

$\mathbb{P}(A) = \mu^{\mathbb{N}}(\phi^{-1}(A))$.

w_n is a random variable : $w_n : (G, \mu)^{\mathbb{N}} \to G$
$(g_i) \mapsto g_1 \cdots g_n$.

$w_0 = 1$

$\mathbb{P}(w_1 = g) = \mu(g)$. w_1 has dist given by μ.

$\mathbb{P}(w_2 = g) = \sum_{h \in G} \mu(h) \mu(h^{-1} g) \stackrel{\text{Defn}}{=} \mu * \mu(g)$ <u>convolution</u> of μ with itself

similarly dist. of w_n given by $\mu_n = \underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}}$

<u>Observation</u> w_n, w_{n+1} <u>not</u> independent.

special case of a <u>Markov chain</u>. set. $\mathcal{U}$, transition probs. $p(u, v)$. Markov property: w_{n+1} depends only on location w_n.

<u>Key example</u> simple r.w. on F_2

recall: r.w. is <u>transient</u>, only hits 1 finitely many times.

$\Rightarrow$ transient on any compact / bounded set.

∂F_2 = space of ends = space of geodesic rays.

transience $\Rightarrow$ a.e. sample path converges to ∂F_2.

so. r.w on F_2 gives hitting measure on boundary $(F_2, \mu) \rightsquigarrow (\partial F_2, \nu)$.

<u>Fact</u>: ∂F_2 uncountable, ν non-atomic, make linear progress in F_2.

<u>Aim</u> generalize this to other groups.

Gromov hyperbolic spaces / groups

(X,d) metric space : X set. $d: X\times X \to \mathbb{R}$ $\quad d(x,x)=0$ ~~non-degenerate~~
$\quad d(x,y)=d(y,x)$ symmetric
$\quad d(x,y)+d(y,z)\geq d(x,z)$ triangle inequality

X proper / locally compact if $B(x,r)$ compact for all x,r.

Examples. $\mathbb{Z}, \mathbb{R}, \mathbb{R}^n$, hyperbolic space, graphs w/ induced path metric w/ every edge length $=1$.

Defn (X,d) is geodesic if ~~for every~~ every pair of points (x,y) is connected by a path of length $d(x,y)$

Example (not geodesic). $\mathbb{Z} \subseteq \mathbb{R}$ w/ induced metric.

◁

Defn a triangle in a geodesic space is a union of 3 geodesics w/ common endpoints

Defn ~~(X,d) has~~ a triangle is δ-slim if any side is contained in a δ-nbd of the other two

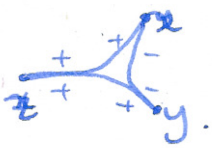
Defn (X,d) has δ-slim triangles if there is a δ which works for all triangles. Then we say (X,d) is Gromov hyperbolic.

Examples. $\mathbb{R}$, $\mathbb{H}^n$ ⊂ surface gps., trees. Non-examples $\mathbb{R}^2, \mathbb{R}^n$ $n\geq 2$.

Alternatively (works better for non-geodesic spaces).

Defn The Gromov product of three points is $(x,y)_z = \frac{1}{2}\left(d(z,x)+d(z,y)-d(x,y)\right)$.

Observation: if (X,d) has δ-slim triangles:



exercise
$(x,y)_z = d(z,[x,y]) \pm O(\delta)$.

some number that depends only on δ.

Defn (X,d) is Gromov hyperbolic if $(x,y)_z \geq \min\{(x,w)_z, (y,w)_z\}-\delta$ for all $x,y,z,w\in X$

Fact (Thm) two defns are equivalent.

Defn a group G is Gromov hyperbolic if $\mathrm{Cay}(G,S)$ is Gromov hyperbolic.

Examples $\mathbb{Z}$, F_2.

Q: is this well defined?

Defn A map $f:(X,d)\to(Y,d)$ is a quasi-isometric embedding if $\exists K,c$ s.t. $\frac{1}{K}d(x,y)-c \leq d(f(x),f(y)) \leq Kd(x,y)+c$

i.e. f distorts distance by a bounded amount on a large scale.

f is a quasi-isometry if f is coarsely onto i.e. $\exists K'$ s.t. $N_{K'}(f(X)) = Y$.

Examples $\mathbb{Z} \underset{QI}{\cong} \mathbb{R}$. $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$, $n \mapsto (n,0)$ is a quasi-isometric embedding

Fact If (X,d) is Gromov hyp. then any space QI to X is Gromov. hyp.

Corollary $\mathrm{Cay}(G,S) \underset{QI}{\cong} \mathrm{Cay}(G,S')$ (S, S' both finite) $\Rightarrow$ G Gromov hyperbolic is well defined.