Math 871 Random walks on geometric groups

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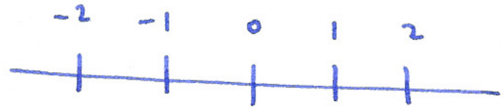
Basic examples

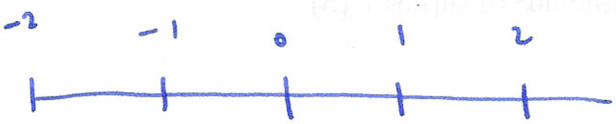
Nearest neighbour/simple random walk on $\mathbb{Z}$



- start at 0 at time $n=0$
- at each subsequent time go right or left with equal probability.

notation w_n = location at time n. then

- $w_0 = 0$
- $w_{n+1} = \begin{cases} w_n + 1 & \text{w/prob } 1/2 \\ w_n - 1 & \text{w/prob } 1/2 . \end{cases}$



			1					$n=0$
		1/2		1/2				$n=1$
	1/4		2/4		1/4			$n=2$
1/8		3/8		3/8		1/8		$n=3$
						⋮		

explicitly

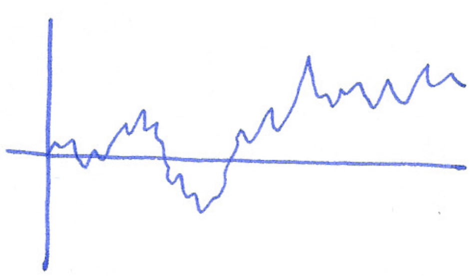
$$\mathbb{P}(w_n = n-2k) = \frac{1}{2^n}\binom{n}{k}$$

Alternate view.

steps: $g_1, g_2, g_3 \ldots$

indep. id. dist r.v.s. $g_i = \begin{cases} +1 & \text{w/p } 1/2 \\ -1 & 1/2 \end{cases}$

$w_n = g_1 + g_2 + \cdots + g_n$.



← a particular sample path

Q: what's $\mathbb{P}(w_n = 0)$?

A: $\mathbb{P}(w_{2n} = 0) = \frac{1}{2^{2n}}\binom{2n}{n}$ Stirling's approximation: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$= \frac{1}{2^{2n}} \frac{(2n)!}{n!\,n!} = \frac{1}{2^{2n}} \frac{\sqrt{4\pi n}\cdot \frac{(2n)^{2n}}{e^{2n}}}{2\pi n \; n^{2n}/e^{2n}} = \frac{2\sqrt{\pi}\sqrt{n}}{2\pi n} = \frac{1}{\sqrt{\pi n}}$$

notation: $\mathbb{P}(w_{2n}=0) \sim \frac{1}{\sqrt{n}}$ (asymptotic up to constant)

Q: how often do you hit zero?

average/expected number of times is $\mathbb{E}(\# w_n = 0) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} = \infty$.

A: hit 0 infinitely often.

Def The simple random walk on $\mathbb{Z}$ is <u>recurrent</u>.

Q: how far away from 0 at time n?

wrong answer: $\mathbb{E}(w_n) = 0$ by symmetry

better answer: $\mathbb{E}(|w_n|)$ easier: $\mathbb{E}(w_n^2) = \cancel{\mathbb{E}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j\right)}$

use: $w_n = \sum_{i=1}^{n} g_i$, so $\mathbb{E}(w_n^2) = \mathbb{E}\left(\left(\sum_{i=1}^{n} g_i\right)^2\right) = \mathbb{E}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} g_i g_j\right)$.

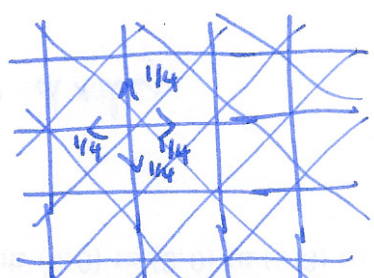
$$= \mathbb{E}\left(\sum_{i=1}^{n} g_i^2 + \sum_{i\neq j} w_i w_j\right) = \mathbb{E}\left(\underbrace{\sum_{i=1}^{n} (\pm 1)^2}_{n} + \sum_{i \neq j} w_i w_j\right) = n + \underbrace{\sum_{i\neq j} \mathbb{E}(w_i w_j)}_{0}$$

$$\begin{array}{c|cc} & w_i = -1 & w_i = +1 \\ \hline w_j = -1 & +1 & -1 \\ w_j = + & -1 & +1 \end{array}$$

$\mathbb{E}(w_n^2) = n$ **Fact** $\mathbb{E}(|w_n|) \sim \sqrt{n}$.

Observation: recurrent, but $|w_n| \sim \sqrt{n}$, sample paths rescale to fractals.

Example $\mathbb{Z}^2$



$g_i = \{(1,0), (-1,0), (0,1), (0,-1)\}$ w/ equal prob.

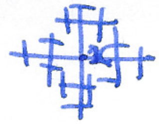
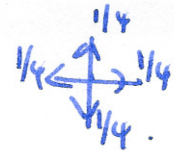
$w_n = \sum_{i=1}^{n} g_i$

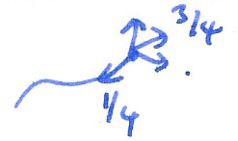
Observation: w_n is the result of two <u>independent</u> r.w.s on lines of slope ± 1

Corollary: $\mathbb{P}(w_n = (0,0)) \sim \frac{1}{n}$.

Fact $\mathbb{E}(|w_n|) \sim \sqrt{n}$.

Example $\mathbb{Z}^d$ **Facts** $\mathbb{P}(w_n = \underline{0}) \sim \frac{1}{n^{d/2}}$ $\mathbb{E}(|w_n|) \sim \sqrt{n}$.

Example 4-valent tree.  

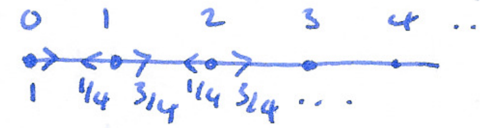


define $|w_n| = d(x_0, w_n)$

$\mathbb{E}(|w_1|) = 1$ $\quad \mathbb{E}(|w_{n+1}|) \geqslant \mathbb{E}(|w_n|) + \frac{1}{2}$. in particular $\mathbb{E}(|w_n|) \geqslant \frac{1}{2}n$.

Q: is this transient? (i.e. not recurrent).

Can think of distance as given by this Markov chain:



claim $\exists\, c < 1$ s.t. $\mathbb{P}(|w_n| = 0) \leq c^n$.

Pf let $p_n = \mathbb{P}$(you ever hit x_0 starting from $|w_n| = n$). so $p_0 = 1$.

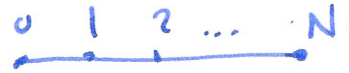
$$p_n = \frac{1}{4} p_{n-1} + \frac{3}{4} p_{n+1}.$$

look for sol^n $p_n = \alpha^n$: $\quad \alpha^n = \frac{1}{4}\alpha^{n-1} + \frac{3}{4}\alpha^{n+1}$

$$\alpha^{n-1}(1 - 4\alpha + 3\alpha^2) = \alpha^{n-1}(\alpha - 1)(3\alpha - 1)$$

so $p_n = A + B(1/3)^n$ for some constants A, B.

Consider finite system with absorbing barrier at N:



$p_0 = 1$, $p_N = 0$ give $\left.\begin{array}{l} p_1 = 1 = A + B \\ p_N = 0 = A + B(1/3)^N \end{array}\right\}$ $A = \frac{-1}{3^N - 1}$ $\quad B = \frac{3^N}{3^N - 1}$.

take limit as $N \to \infty$ gives $p_n = (1/3)^n$, decays exp in n. ∎.

~~$\mathbb{E}(\#\text{ times at } |w_n| = 0) = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots = \frac{1}{1 - 1/3} = 3/2 < \infty$~~.

from 1, 2/3 of paths never hit 0, those that hit 0, ~~eventually~~ next hit 1, by the Markov property, 2/3 of them never hit zero...

so $\mathbb{P}(|w_{2n}| = 0) \leq (1/3)^n$, decays exp in n as required □.

in particular, $\mathbb{E}(\#\, |w_n| = 0) \leq \sum_{n=0}^{\infty} (1/3)^n = \frac{1}{1 - 1/3} = 2/3 < \infty$.

Remark secretly, these graphs are Cayley graphs of groups. $\mathbb{Z}, \mathbb{Z}^2, F_2$, etc..

Group: $\langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$. gen set S. Cayley(G, S) = graph w/ vertices $g \in G$
edges connect (g, gs).