Math 505 Introduction to Proofs Spring 19 Midterm 2

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a 3 × 5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) (a) Prove that for all integers x, if x^3 is even, then x is even.
(b) Prove that $\sqrt[3]{2}$ is irrational. (You may use part (a).)

a) Thm If x^3 is even, then x is even.
Proof (contrapositive) Suppose x is odd. Then $x = 2k+1$ for some $k \in \mathbb{Z}$.
Then $x^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$ odd,
so x odd $\Rightarrow x^3$ odd, as required. $\square$

b) Thm $\sqrt[3]{2}$ is irrational.
Proof (contradiction) Suppose $\sqrt[3]{2} = \frac{a}{b}$, where a, b are integers w/ no common factor. Then $2 = \frac{a^3}{b^3}$, so $2b^3 = a^3$, so a^3 is even. a) $\Rightarrow$ a is even, so $a = 2k$ for some $k \in \mathbb{Z}$, so $2b^3 = (2k)^3 = 8k^3$. Therefore $b^3 = 4k^3$, so b is also even, so a, b have a common factor. # $\square$

(2) Give examples of:

(a) a function $f\colon \mathbb{R} \to \mathbb{R}$ which is injective but not surjective.

(b) a function $f\colon \mathbb{N} \to \mathbb{Z}$ which is surjective but not injective.

a) $f(x) = e^x$ check: if $e^x = e^y$ then $x = y$ so injective.

$e^x \geq 0$ so not surjective.

b)

x	1	2	3	4	5	6	7	8	9 ...
$f(x)$	0	0	1	-1	+2	-2	+3	-3	+4 ...

equivalently $f(x) = \begin{cases} -\frac{x}{2} + 1 & \text{if } x \text{ even} \\ \frac{x-1}{2} & \text{if } x \text{ odd.} \end{cases}$

(3) Give an explicit bijection between the interval $(0, 1)$ and the interval $(0, \infty)$.

$f(x) = \frac{1}{x}$ maps $(0,1)$ to $(1,\infty)$ injectively

$g(x) = x-1$ maps $(1,\infty)$ to $(0,\infty)$ injectively

so $g(f(x)) = \frac{1}{x} - 1$ maps $(0,1)$ to $(0,\infty)$ injectively

(4) State the negation of the following statement, using appropriate quantifiers:
The function $f: A \to B$ is a bijection. $\circledast$

$\circledast$: $(\forall b \in B)(\exists a \in A)(f(a) = b)$ and $\cancel{(\forall b \in B)}(\forall a_1, a_2 \in A)(f(a_1) = f(a_2) \Rightarrow a_1 = a_2)$

not $\circledast$: $(\exists b \in B)(\forall a \in A)(f(a) \neq b)$ or $(\exists a_1, a_2 \in A)(f(a_1) = f(a_2)$ and $a_1 \neq a_2)$

(5) State the negation of the following statement, using appropriate quantifiers:
(*) The function $f\colon \mathbb{R} \to \mathbb{R}$ is bounded above.

(*) : $(\exists B \in \mathbb{R})(\forall x \in \mathbb{R})(f(x) \leq B)$.

not (*) : $(\forall B \in \mathbb{R})(\exists x \in \mathbb{R})(f(x) > B)$

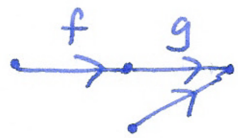
(6) Write out a careful proof or give a counterexample to the following statement:
For any integer, the sum of the integer with its square is even.

Thm: n^2+n is even

Proof $n^2+n = n(n+1)$, but for any two consecutive numbers, at least one of them is even. Claim: if $a \in \mathbb{Z}$ is even, then for any $b \in \mathbb{Z}$, ab is even. Proof (of claim) If $2|a$ then $a = 2c$ for some $c \in \mathbb{Z}$, so $ab = 2cb$, even. $\square$. Claim $\Rightarrow$ $n(n+1)$ is even as required. $\square$

(7) Write out a careful proof or give a counterexample to the following statement:
If $g \circ f$ is surjective then f is surjective.

counter example :



$g \circ f$ surjective

f not surjective

(8) Write out a careful proof or give a counterexample to the following statement:
The composition of increasing functions is increasing, where we say a function $f\colon \mathbb{R} \to \mathbb{R}$ is increasing if $x \leqslant y$ implies $f(x) \leqslant f(y)$.

Thm: The composition of increasing functions is increasing

Proof Let f and g be increasing functions, i.e.

$x \leq y \Rightarrow f(x) \leq f(y)$

and $x \leq y \Rightarrow g(x) \leq g(y)$

suppose $x \leq y$ then $f(x) \leq f(y)$ as f increasing.

then as g increasing $f(x) \leq f(y) \Rightarrow g(f(x)) \leq g(f(y))$

so $g \circ f$ is increasing, as required. $\square$

(9) State the contrapositive and converse of the following statement, and then prove or give a counterexample to each statement:

If $f\colon \mathbb{R} \to \mathbb{R}$ is strictly decreasing then it is injective.

contrapositive: if f is not injective then it is not strictly decreasing

Proof If f not injective, then $\exists\, x, y \in \mathbb{R}$, $x \neq y$ but $f(x) = f(y)$ wlog $x < y$, but then $x < y$ and $f(x) = f(y)$ # strictly decreasing $\square$.

converse: if f is injective then f is strictly decreasing

counterexample: $f(x) = x$ injective, but not strictly decreasing

(10) Let $f\colon X \to Y$ be an injective function function. Prove that for any $A, B \subseteq X$, $f(A \cap B) = f(A) \cap f(B)$.

Thm $\forall A, B \subseteq X$, $f(A \cap B) = f(A) \cap f(B)$

Proof $\subseteq$ suppose $x \in f(A \cap B)$, then $\exists$ $y \in A \cap B$ s.t. $f(y) = x$

as $y \in A$, $x = f(y) \in f(A)$

as $y \in B$ $x = f(y) \in f(B)$ so $x = f(y) \in f(A) \cap f(B)$, as required.

$\supseteq$ spose $x \in f(A) \cap f(B)$

as $x \in f(A)$ there is $y \in A$ s.t. $f(y) = x$

as $x \in f(B)$ there is $\cancel{y}\, z \in B$ s.t. $f(z) = x$

f injective $\Rightarrow$ if $f(y) = f(z)$ then $y \cancel{x} = \cancel{y}\, z$, so in fact

$y = z \in A$ and B, so lies in $A \cap B$, as required. $\square$.