**Math 505 Introduction to Proofs Spring 19 Sample Midterm 2**

(1) Show that $\sqrt[3]{3}$ is irrational.

(2) Find an example of a function $f\colon \mathbb{N} \to \mathbb{N}$ which is
  (a) injective but not surjective
  (b) neither injective nor surjective

(3) Find an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is
  (a) surjective but not injective
  (b) neither injective nor surjective

(4) Find explicit bijections between the following subintervals of $\mathbb{R}$.
  (a) $(0,1)$ and $(1,\infty)$
  (b) $(1,\infty)$ and $(0,\infty)$
  (c) $(0,\infty)$ and $\mathbb{R}$
  (d) $(0,1)$ and $\mathbb{R}$

(5) Say a set $A$ is *countable* if there is an injective map $f\colon A \to \mathbb{N}$. Show that the product of countable sets is countable. Show that the set of $2 \times 2$ matrices with integer coefficients is countable.

(6) What do the following statements mean in ordinary language?
  (a) $(\exists n \in N)(x = 2^n)$
  (b) $(\forall a, b \in \mathbb{N})(\pi \neq a/b)$
  (c) $(\forall b \in B)(\exists a \in A)(f(a) \neq b)$
  (d) $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(n^2 + n = 2m)$
  (e) $(\exists A \subset \mathbb{R})(\forall x \in \mathbb{R})(\exists a \in A)(x < a)$
  (f) $(\forall n \in \mathbb{N})(\exists p \in \mathbb{N})((p > n)$ and $[(\forall q \in \mathbb{N})(q|p \Rightarrow (q = 1)$ or q = p)])

(7) State the negation of the following statements, using appropriate quantifiers.
  (a) $e^\pi$ is rational.
  (b) The function $f$ is surjective but not injective.
  (c) The integer $n$ is divisible by a square number.
  (d) There is a surjective function $f\colon A \to B$.
  (e) The function $f\colon \mathbb{R} \to \mathbb{R}$ is bounded above and below.
  (f) $\lim_{x\to 0} f(x) = 1$.

(8) Write out careful proofs, or give counterexamples, to the following statements.
  (a) Show that the product of any three consecutive integers is divisible by 6.

(b) Show that the sum of any three consecutive integers is divisible by 3.
(c) A function $f\colon A \to B$ has an inverse if and only if it is both surjective and injective.
(d) If $f$ is surjective, then $f \circ g$ is surjective.
(e) If $f$ and $f \circ g$ are injective, then $g$ is injective.
(f) If $f \circ g\colon \mathbb{R} \to \mathbb{R}$ is increasing, and $g\colon \mathbb{R} \to \mathbb{R}$ is increasing, then $f$ is increasing.
(g) If $x \in \mathbb{R}$ and $x^2 \geq x$, then $x \geq 1$.

(9) Let $f\colon X \to Y$ be a function. If $A \subseteq X$, let $f(A)$ be the image of $A$ in $Y$. Show that this defines a function from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$. Can you say when it is injective or surjective?

(10) Suppose that $f\colon A \to B$ and let $C \subseteq B$.
(a) Prove or give a counterexample: $f^{-1}(B - C) \subseteq f^{-1}(B) - f^{-1}(C)$
(b) Prove or give a counterexample: $f^{-1}(B) - f^{-1}(C) \subseteq f^{-1}(B - C)$
(c) Is there a condition on $f$ that will ensure that $f^{-1}(B - C) = f^{-1}(B) - f^{-1}(C)$? Explain.
(d) Is there a condition on $f$ that will ensure that $f^{-1}(B - C) = A - f^{-1}(C)$? Explain.