SMT2 Solutions

Q1 **Thm** $\sqrt[3]{3}$ is irrational.

Proof suppose not, then $\sqrt[3]{3} = a/b$ with a,b no common factor $\Rightarrow 3 = \frac{a^3}{b^3} \Rightarrow 3b^3 = a^3$, so $3 \mid a^3$

3 prime, so $3 \mid a^3 \Rightarrow 3 \mid a$, so $a = 3c$ for some $c \in \mathbb{Z}$. $3b^3 = (3c)^3 = 27c^3$

so $b^3 = 9c^3 \Rightarrow 3 \mid b$ # a,b in lowest terms $\square$.

Q2. a) $f: \mathbb{N} \to \mathbb{N}$ $f(n) = n+1$

b) $f: \mathbb{N} \to \mathbb{N}$ $f(n) = \begin{cases} 1 & \text{if } n = 1,2 \\ 2 & \text{otherwise} \end{cases}$

Q3 a) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} x & x \leq 0 \\ 0 & 0 \leq x \leq 1 \\ x-1 & x \geq 1 \end{cases}$. b) $f(x) = \sin(x)$.

Q4 a) $(0,1) \to (1,\infty)$, $x \mapsto 1/x$ b) $(1,\infty) \to (0,\infty)$, $x \mapsto x-1$ c) $(0,\infty) \to \mathbb{R}$, $x \mapsto \log(x)$ d) $(0,1) \to \mathbb{R}$, $x \mapsto \log(\frac{1}{x} - 1)$.

Q5 show $\mathbb{N} \times \mathbb{N}$ countable:

	1	2	3	4 ...
1	(1,1)	(1,2)	(1,3)	
2	(2,1)	(2,2)		
3	(3,1)			
4				
⋮				

$f: A \to \mathbb{N}$, $g: B \to \mathbb{N}$ injective

then $(f \times g): A \times B \to \mathbb{N} \times \mathbb{N}$ injective.
$(a,b) \mapsto (f(a), g(b))$

so $A \times B \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ injective.

Q6 a) x is a power of 2

b) π is irrational

c) f is not constant function

d) n^2+n is even

~~d) A is not bounded above~~ there is an

e) there is a subset $A \subset \mathbb{R}$ which is not bounded above

f) the set of primes is not bounded above.

Q7 a) $(\exists a,b \in \mathbb{N})(e^\pi = a/b)$ negation: $(\forall a,b \in \mathbb{N})(e^\pi \neq a/b)$.

b) $(\forall b \in B)(\exists a \in A)(f(a) = b)$ and $(\exists a,b \in A)(a \neq b \text{ and } f(a) = f(b))$

negation: $(\exists b \in B)(\forall a \in A)(f(a) \neq b)$ or $(\forall a,b \in A)(a = b \text{ or } f(a) \neq f(b))$.

c) $(\exists k \in \mathbb{N})(n = k^2)$ negation $(\forall k \in \mathbb{N})(n \neq k^2)$.

d) $(\exists R_f \subset A \times B)\big((\forall a \in A)(\exists b \in B)((a,b) \in R_f)$ and $(\forall a \in A)(\forall b,c \in B)((a,b) \in R_f \text{ and } (a,c) \in R_f \Rightarrow b = c)$ and $(\forall b \in B)(\exists a \in A)(f(a) = b) \big)$.

negation: $(\forall R_f \subset A \times B)\big((\exists a \in A)(\forall b \in B)((a,b) \notin R_f)$ or $(\exists a \in A)(\exists b,c \in B)((a,b) \in R_f \text{ and } (a,c) \in R_f \text{ and } b \neq c)$ or $(\exists b \in B)(\forall a \in A)(f(a) \neq b) \big)$

c) $(\exists a \in \mathbb{R})(\forall x \in \mathbb{R})(f(x) < a)$ and $(\exists b \in \mathbb{R})(\forall x \in \mathbb{R})(f(x) > b)$.

negation: $(\forall a \in \mathbb{R})(\exists x \in \mathbb{R})(f(x) \geq a)$ or $(\forall b \in \mathbb{R})(\exists x \in \mathbb{R})(f(x) \leq b)$.

f) $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(|x| \leq \delta \Rightarrow |f(x) - 1| \leq \epsilon)$.

negation: $(\exists \epsilon > 0)(\forall \delta > 0)(\exists x \in \mathbb{R})(|x| \leq \delta$ and $|f(x) - 1| > \epsilon)$.

Q8 a) Thm The product of any three consecutive integers is divisible by 6

Proof Let $n, n+1, n+2$ be three consecutive integers. ~~Their product is $n(n+1)(n+2) = n^3+3n^2+2n$~~
at least one must be even, at least one must be divisible by 3. So if $p = n(n+1)(n+2)$
$2|p$ and $3|p \Rightarrow 6|p$ as 2,3 coprime. □.

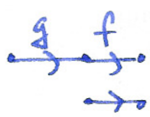
b) Thm The sum of any three consecutive integers is divisible by 3.

Proof Let $n, n+1, n+2$ be three consecutive integers. Then their sum is $n+n+1+n+2 = 3n+3$
$= 3(n+1)$, which is divisible by 3 □.

c) Thm $f: A \to B$ has an inverse iff it is surjective and injective.

Proof $\Rightarrow$ Suppose f has an inverse function $f^{-1}: B \to A$. Then for all $b \in B$. $f^{-1}(b) \in A$, with
$f(f^{-1}(b)) = b$, so f is surjective. Suppose $f(a) = f(b)$. then $f^{-1}(f(a)) = f^{-1}(f(b)) \Rightarrow a = b$, so
f is injective.

$\Leftarrow$ If f is surjective, then for any $b \in B$ $\exists a \in A$ s.t. $f(a) = b$. In particular $f^{-1}(\{b\}) \neq \emptyset$.
Now suppose $a, a' \in f^{-1}(b)$, then $f(a) = f(a') = b$, as f injective $\Rightarrow a = a'$, so $f^{-1}(\{b\})$
contains a single element. So define $f^{-1}: B \to A$ to send b to the unique element in
$f^{-1}(b)$. Check this is a function: $\forall b \in B$, $f^{-1}(b) \neq \emptyset$, so there is a pair $(b,a) \in R_f$
$\subseteq B \times A$. If (b,a) and (b,a') lie in $R_f \subseteq B \times A$ then a and a' lie in $f^{-1}(b)$, so are
equal. □.

d) false:  f surjective but $f \circ g$ not surjective.

e) Thm If f and $f \circ g$ are injective, then g is injective.

Proof (contrapositive) suppose g is not injective, then there are elements $a \neq b$ s.t. $g(a) = g(b)$
but then $f(g(a)) = f(g(b)) \Rightarrow f \circ g$ is not injective. □.

f) false: $f(x) = x^2$, $g(x) = e^x$, $f(g(x)) = e^{2x}$ increasing
not increasing, increasing

g) false: $x = 0$.

Q9 $f: X \to Y$ defines a function $\Psi: P(X) \to P(Y)$
$A \mapsto f(A)$

Suppose Ψ is surjective, the for every $U \in P(Y)$, $\exists\ V$ s.t. $\Psi(V) = U$, i.e. for every $U \subseteq Y$ $\exists\ V$ s.t. $f(V) = U$. Choose $U = Y$, then $\exists\ V \subseteq X$ s.t. $f(V) = Y \Rightarrow f$ is surjective.

Suppose Ψ is injective, the $\Psi(U) = \Psi(V) \Rightarrow U = V$. Consider the one element sets $\{x\}$ and $\{y\}$. Suppose $\Psi(\{x\}) = \Psi(\{y\})$ then $f(\{x\}) = f(\{y\})$ so $\{f(x)\} = \{f(y)\}$ $\Rightarrow f(x) = f(y)$. but then Ψ injective $\Rightarrow$ $\{x\} = \{y\} \Rightarrow x = y$, so f is injective. $\square$.

Q10 $f: A \to B$ and $C \subseteq B$.

a) Thm $f^{-1}(B-C) \subseteq f^{-1}(B) - f^{-1}(C)$

Proof note that $f^{-1}(B) = A$, so suffices to show $f^{-1}(B-C) \subseteq A - f^{-1}(C)$

suppose $x \in f^{-1}(B-C)$, then $f(x) \in B - C$, in particular $f(x) \notin C$, so $x \notin f^{-1}(C)$

so $x \in f^{-1}(C)' = A - f^{-1}(C)$ $\square$.

b) Thm $f^{-1}(B) - f^{-1}(C) \subseteq f^{-1}(B-C)$

Proof let $x \in f^{-1}(B) - f^{-1}(C)$. Then $f(x) \in B$ and $f(x) \notin C$, so $f(x) \in B - C$ but then $x \in f^{-1}(B-C)$ $\square$.

c) we've shown this always happens.

d) $f^{-1}(B) = A$, so this always happens.