Math 505 Introduction to Proofs Spring 19 Midterm 1

Name: Solutions

- **Start each question on a fresh sheet of paper. Staple together in numerical order at the end of the exam.**

(1) For each of the following statements, find two distinct elements in the truth set, and two distinct elements not in the truth set. (Indicate clearly which are which.)

(a) a is an integer divisible by either 2 or 3.

(b) A is an infinite subset of $\mathbb{Z}$.

(2) Consider the statement:

If x and y are real numbers with $x < y$ then $x^2 < y^2$.

Which, if any, of the following substitutions give a counter example.

(a) $x = 1, y = 2$ (b) $x = 1, y = 0$ (c) $x = -2, y = -1$

(3) Write out a careful proof of the fact that the product of any two odd numbers is odd.

(4) Prove or disprove the following statement: If $A \cup B = A \cup C$ then $B = C$.

(5) Prove or disprove the following statement: $A - (B - C) = (A - B) \cup C$.

(6) Prove or disprove the following statement: $A \cap B' = A - B$.

(7) State which of the following statements, are true, vacuously true, or false.

(a) For integers a, b and c, if $a \mid b$ and $b \mid c$ then $a \mid b + c$.

(b) If x is a real number with $x^2 < 0$ then $x < 0$.

(c) If $A \cup B \subseteq C \cup D$ then $A \cap B \subseteq C \cap D$.

(8) Suppose A and B are finite sets with $|A| = a$, $|B| = B$ and $|A \cap B| = c$. Find $|\mathcal{P}(A \cup B)|$.

(9) Write out a careful proof of the fact that for integers a and b, if $a \mid b$ then $a^2 \mid b^2$.

(10) Write out a careful proof of the fact that if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Q1 truth set not in truth set

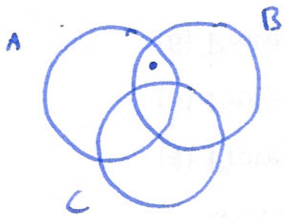
a) 2, 3 5, 7

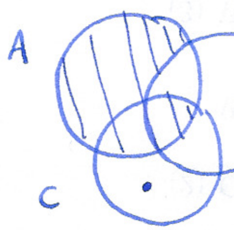
b) $\mathbb{Z}$, $\mathbb{N}$ $\emptyset$, $\{0\}$

Q2 c) counterexample

Q3 **Thm** The product of any two odd numbers is odd.

Proof Let x and y be odd integers. Then $x = 2m+1$ and $y = 2n+1$ for some integers m, n. Then $xy = (2m+1)(2n+1) = 4mn + 2m + 2n + 1$ $= 2(2mn + m + n) + 1$, which is odd $\square$.

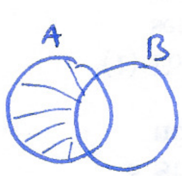
Q4  false, choose $A = B = \{1\}$, $C = \emptyset$.

Q5 

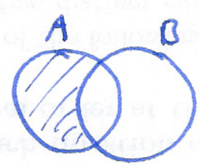
$A - (B - C)$

$(A - B) \cup C$

false: choose $A = B = \emptyset$, $C = \{1\}$

Q6 

$A \cap B'$

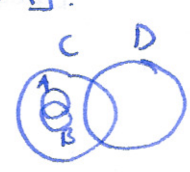


$A - B$

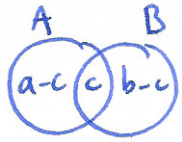
Thm $A \cap B' = A - B$

Proof Suppose $x \in A \cap B'$, then $x \in A$ and $x \in B'$, so $x \in A$ and $x \notin B$ $\Rightarrow$ $x \in A - B$. So $A \cap B' \subseteq A - B$. Now suppose $x \in A - B$, so $x \in A$, but $x \notin B$, so $x \in A$ and $x \in B'$ $\Rightarrow$ $x \in A \cap B'$, so $A - B \subseteq A \cap B'$. Therefore $A \cap B' = A - B$ $\square$.

Q7 a) true b) vacuously true c) false



Q8



$|P(A \cup B)| = 2^{a+b-c}$

$|A \cup B| = a+b-c$

Q9 **Thm** If $a|b$ then $a^2|b^2$

Proof Suppose $a|b$, then $b = an$ for some integer n.

Then $b^2 = a^2n^2$, so $a^2|b^2$, as required $\square$.

Q10 **Thm** If $A \subseteq B$ then $P(A) \subseteq P(B)$

Proof Assume $A \subseteq B$, and suppose $C \in P(A)$, then $C \subseteq A$, but then $C \subseteq A \subseteq B$ by hypothesis of $C \subseteq B$, which implies $C \in P(B)$, so $P(A) \subseteq P(B)$ $\square$.