Math 505 Introduction to Proofs Spring 19 Final

Name: Solutions

- You may use a 3×5 index card of notes, and a calculator.
- I will count your best 10 of the following 12 questions.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

1. Write out a detailed proof of the fact that the sum of any two odd numbers is even.

Thm The sum of any two odd numbers is even

Proof Let a and b be odd numbers, so $a = 2m+1$, $b = 2n+1$ for integers m, n.

Then $a+b = 2m+1 + 2n+1 = 2(m+n+1)$ which is even. $\square$.

2. (a) Show that for any integer n, if n^2 is even, then n is even.
   (b) Use this to show that $\sqrt{2}$ is irrational.

a) Thm If n^2 is even, then n is even. (*)

Proof (contrapositive) Suppose n is odd. Then $n = (2m+1)$ for some integer m.

so $n^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2+2m)+1$, odd $\square$

b) Thm $\sqrt{2}$ is irrational

Proof (contradiction) Suppose $\sqrt{2} = \frac{a}{b}$, a, b integers w/ no common factor.

Then $2 = \frac{a^2}{b^2}$, so $2b^2 = a^2$, so a^2 is even. (*) $\Rightarrow$ a is even

so $a = 2k$ for some integer k. Then $2b^2 = (2k)^2 = 4k^2$

$\Rightarrow$ $b^2 = 2k^2$ $\Rightarrow$ b is even # no common factor $\square$

3. Consider the statement:

   If A and B are infinite subsets of $\mathbb{N}$, then $A \cap B$ is infinite.

   Which, if any, of the following substitutions is a counterexample?

   (a) $A =$ odd numbers, $B =$ prime numbers.

   (b) $A = \{1, 2, \ldots, n\}$, $B = \{n, n+1, \ldots, \}$.

   (c) $A =$ prime numbers, $B =$ even numbers.

c) is the only counterexample

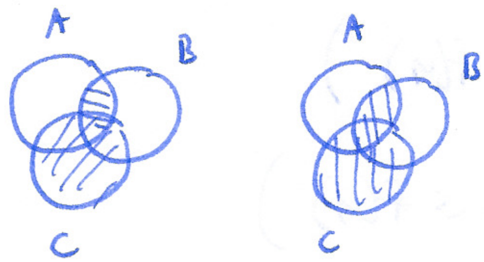
4. Indicate which of the following statements are true, vacuously true, or false.

(a) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(b) If x is a real solution to $x^2 + 2x + 2 = 0$ then $x < 0$.

(c) If $\mathcal{P}(A) \neq \varnothing$, then $A \neq \varnothing$.

a)



true

b) $x^2 + 2x + 2 = (x+1)^2 + 1$ no real solutions, vacuously true.

c) false (counterexample $A = \phi$).

5. Write out the following statements using quantifiers. Then write out their negations.

   (a) The function $f\colon \mathbb{R} \to \mathbb{R}$ is strictly decreasing.

   (b) The set $A \subseteq \mathbb{N}$ is bounded above.

a) $(\forall x, y \in \mathbb{R})\,(x < y \Rightarrow f(x) > f(y))$

negation: $(\exists x, y \in \mathbb{R})\,(x < y \text{ and } f(x) \leq f(y))$

b) $(\exists N \in \mathbb{N})$ s.t. $(\forall a \in A)\,(a \leq N)$

negation: $(\forall N \in \mathbb{N})\,(\exists a \in A)\,(a > N)$

6. (a) Write out what it means for a function $f\colon A \to B$ to be injective.
(b) Show that the composition of two injective functions is injective.

a) $f: A \to B$ injective means $(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$

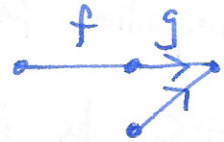
b) Thm: The composition of two injective functions is injective

Proof Let $f: A \to B$ and $g: B \to C$ be injective, and let $x, y \in A$. Suppose $g(f(x)) = g(f(y))$. As g is injective, this implies $f(x) = f(y)$. As f is injective, this implies $x = y$. Hence $g \circ f$ is injective. $\square$

7. State the converse of "The composition of surjective functions is surjective", then prove or give a counterexample to the converse.

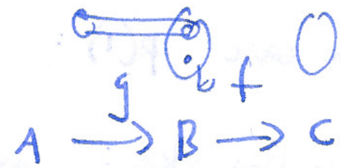
Converse: Let $f: A \to B$ and $g: B \to C$. If $g \circ f$ is surjective, then f and g are surjective.

False. counterexample:



8. State the contrapositive of "If f is injective and $f \circ g$ is surjective then g is surjective. Proof or give a counterexample.

Contrapositive: If g is not surjective, then either f is not injective, or $f \circ g$ is not surjective.



Proof Let $f : B \to C$ and $g : A \to B$

If g is not surjective, then $\exists\, b \in B$ s.t. $(\forall a \in A)(f(a) \neq b)$.

- if f not surjective, then we are done.
- suppose f is injective, then $f(b) \neq f(a)$ for all $a \in A$ $\Rightarrow$ $f \circ g$ not surjective. $\square$.

~~suppose $f \circ g$ is surjective, then~~

9. Use induction to show that $2 + 4 + 6 + \cdots + 2n = n(n+1)$.

Proof (induction) $p(n)$: $2+4+6+\cdots+2n = n(n+1)$.

base case $p(1)$: $2 = 1 \cdot 2$ ✓

induction step: consider $\underbrace{2+4+6+\cdots+2n}_{= n(n+1) \text{ by } p(n)} + 2(n+1)$

$= n(n+1) + 2n+2 = n^2 + 3n + 2 = (n+1)(n+2)$

so $p(n+1)$ holds. $\square$.

10. Use induction to show that $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

Proof (induction)

base case $p(1)$: $\frac{1}{1\cdot 2}=\frac{1}{2}=\frac{1}{1+1}=\frac{1}{2}$ ✓

induction step : consider $\underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}}_{=\frac{n}{n+1}\text{ by } p(n)}+\frac{1}{(n+1)(n+2)}$

$$=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}=\frac{1}{n+1}\left(n+\frac{1}{n+2}\right)=\frac{1}{n+1}\left(\frac{n^2+2n+1}{n+2}\right)$$

$$=\frac{(n+1)^2}{(n+1)(n+2)}=\frac{n+1}{n+2}$$ so $p(n+1)$ holds $\Box$.

11. Show that if x is a real number and $x + \frac{1}{x}$ is an integer, then $x^n + \frac{1}{x^n}$ is an integer.

Proof (induction).

base case : $p(1)$: $x + \frac{1}{x}$ is an integer, holds by assumption.

$p(2)$: $(x+\frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2} \Rightarrow x^2 + \frac{1}{x^2} = 2 - n^2 \Rightarrow$ integer.

so $p(2)$ holds.

induction step : assume $p(n)$: $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$.

consider $$\underbrace{\left(x^n + \frac{1}{x^n}\right)}_{\in \mathbb{Z} \text{ by } p(n)} \underbrace{\left(x + \frac{1}{x}\right)}_{\in \mathbb{Z} \text{ by } p(1)} = x^{n+1} + x^{n-1} + \frac{1}{x^{n-1}} + \frac{1}{x^{n+1}}.$$

$$= x^{n+1} + \frac{1}{x^{n+1}} + \underbrace{\left(x^{n-1} + \frac{1}{x^{n-1}}\right)}_{\in \mathbb{Z} \text{ by } p(n-1)}$$

$\Rightarrow$ $x^{n+1} + \dfrac{1}{x^{n+1}} \in \mathbb{Z}$ so $p(n+1)$ holds $\square$.

12. Consider the relation R on $\mathbb{N}$ given by $a \sim b$ if there is a prime number p such that $p \mid a$ and $p \mid b$. Is R an equivalence relation on $\mathbb{N}$? Explain.

No, not reflexive, there is no prime number p s.t. $p \mid 1$

So ~~[illegible]~~ $1 \not\sim 1$.