Math 505 Introduction to Proofs Spring 19 Sample Final

(1) Consider the statement

If x and y are real numbers with $x < y$ then $x/y < 1$.

Which, if any, of the following substitions gives a counterexample:

(a) $x = 1, y = 2$ (b) $x = 2, y = -1$ (c) $x = -2, y = -1$

(2) State which of the following statements are true, vacuously true, or false.

(a) $(A \cap B)' = A' \cap B'$

(b) If $\mathcal{P}(A) = \varnothing$, then $A = \varnothing$

(c) $(A \cup B) - C = (A - C) \cup (B - C)$

(3) Let $f : X \to Y$ be a function, and let A and B be subsets of Y. Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(4) Let $f : X \to Y$ be a function, and let A be a subset of Y. Show that $f(f^{-1}(A)) \subset A$. Is $f(f^{-1}(A)) = A$?

(5) Either give examples of functions with the properties below, or explain why they don't exist.

(a) A surjective function from $\mathbb{N}$ to $[0, 1]$

(b) A surjective function from $(0, \infty)$ to $(0, 1)$

(c) An injective function from $\mathbb{R}^2$ to $\mathbb{R}$

(6) State the negation of the following statements, using appropriate quantifiers.

(a) The function $f : A \to B$ is injective.

(b) There is an injective function $f : A \to B$.

(c) The function $f : \mathbb{R} \to \mathbb{R}$ is strictly decreasing.

(7) Write out careful proofs, or give counterexamples, to the following statements.

(a) If n is an integer then $n^3 - n$ is even.

(b) If g and $g \circ f$ are surjective, then f is surjective.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is increasing, and $g : \mathbb{R} \to \mathbb{R}$ is increasing, then $f + g$ is increasing.

(8) Write out the negation of the statement, "$(\forall x)(p(x))$ or $(\exists x)(\sim q(x))$".

(9) Write out the converse to the statement "If $p(x)$ or $q(x)$ then $r(x)$".

(10) Write out the contrapositive to the statement "If $p(x)$ or $q(x)$ then $r(x)$".

(11) We say a sequence a_n has a limit L if for all $\epsilon > 0$ there is an N such that $|L - a_n| \leq \epsilon$ for all $n \geq N$. Write this statement out using the quantifier symbols, then write out the negation of this statement. Use the negation to show that the sequence $a_n = (-1)^n$ does not have a limit.

(12) Show that the following numbers are irrational: $\sqrt{2}, \sqrt{3}, \sqrt{6}, \sqrt{2} + \sqrt{3}$.

(13) Show that $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + (n)(n+1) = \frac{1}{3}n(n+1)(n+2)$.

(14) Show that $n^2 \leq 2^n$ for $n \geq 5$.

(15) Show that $4^n + 5^n + 6^n$ is divisible by 15 when n is odd.

(16) Let $x_1 = 1$ and $x_{n+1} = \sqrt{1 + 2x_n}$ for $n \geq 1$. Show that $x_n \leq 4$ for all n.

(17) Show that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$.

(18) Consider the relation on $\mathbb{R}$ determined by $x \sim y$ if $x - y$ is irrational. Is this an equivalence relation?

(19) Consider the relation on functions $f\colon \mathbb{R} \to \mathbb{R}$ given by $f \sim g$ if there are numbers $A, B \in \mathbb{R}$ such that $f(x) \leq Ag(x) + B$. Is this an equivalence relation?

(20) Define a relation on sets by $A \sim B$ if there is a bijection between A and B. Is this an equivalence relation on sets? What are the equivalence classes?

(21) Let F be the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$. Define a relation on F by $f \sim g$ if $g = f'$. Is this an equivalence relation? Does it define a function?