MTH 505 SF Solutions

Q1 c) is a counterexample

Q2 a) false b) vacuously true c) true

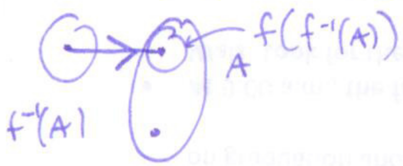
Q3 **Thm** $f: X \to Y$ function, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Proof $\subseteq$: suppose $x \in f^{-1}(A \cup B)$, then $f(x) \in A \cup B$, so $f(x) \in A$ or $f(x) \in B$
then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$ so $x \in f^{-1}(A) \cup f^{-1}(B)$.

$\supseteq$: suppose $x \in f^{-1}(A) \cup f^{-1}(B)$, then $f(x) \in A$ or $f(x) \in B$, so
$f(x) \in A \cup B \Rightarrow x \in f^{-1}(A \cup B)$. $\square$.

Q4 **Thm** $f: X \to Y$, $f(f^{-1}(A)) \subset A$.

Proof suppose $y \in f(f^{-1}(A))$, then $\exists\, x \in f^{-1}(A)$ s.t. $f(x) = y$, but
$x \in f^{-1}(A) \Rightarrow f(x) \in A$, so $f(x) = y \in A$ $\square$.



$f(f^{-1}(A)) \neq A$ in general.

Q5 a) no such function as $[0,1]$ uncountable.

b) $f(x) = \frac{1}{x+1}$ c) send decimals: $(a_n \ldots a_2 a_1 \cdot b_1 b_2 \ldots, c_n \ldots c_2 c_1 \cdot d_1 d_2 \ldots)$ to
$(\ldots c_2 a_2 c_1 a_1 \cdot b_1 d_1 b_2 d_2 \ldots)$

Q6 a) $(\forall x, y)$ If $f(x) = f(y)$ then $x = y$

negation: $\exists\, x, y$ s.t. $f(x) = f(y)$ and $x \neq y$.

b) $(\exists f: X \to Y)\,(\forall x, y)\,(f(x) = f(y) \Rightarrow x = y)$.

negation $(\forall f: X \to Y)\,((\exists x, y)(f(x) = f(y)$ and $x \neq y)$.

c) $(\forall x, y)\,(x < y \Rightarrow f(x) > f(y))$

negation: $(\exists x, y)\,(x < y$ and $f(x) \leq f(y))$.

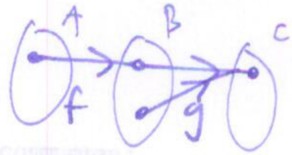
Q7 a) Thm $\forall n \in \mathbb{Z}$, $n^3 - n$ is even

Proof case 1: n is ~~odd~~ even. then $n = 2m$ for some $m \in \mathbb{Z}$.

$n^3 - n = (2m)^3 - (2m) = 8m^3 - 2m = 2(4m^3 - m)$ even.

case 2 n odd. then $n = 2m+1$ for some $m \in \mathbb{Z}$.

$n^3 - n = (2m+1)^3 - (2m+1) = 8m^3 + 12m^2 + 6m + 1 - 2m - 1 = 2(4m^3 + 6m^2 + 2m)$ even

b) counterexample: 

f not surjective.

c) Thm $f, g : \mathbb{R} \to \mathbb{R}$ increasing $\Rightarrow$ $f+g$ increasing

Proof f increasing means $x \geq y \Rightarrow f(x) \geq f(y)$

g increasing means $x > y \Rightarrow g(x) \geq g(y)$

add: $f(x) + g(x) \geq f(y) + g(y)$ (*)

$(f+g)(x) = f(x) + g(x)$, so (*) $\Rightarrow$ $f+g$ increasing. □

Q8 $(\exists x)(\sim p(x))$ and $(\forall x)(q(x))$

Q9 If $r(x)$ then $\sim p(x)$ and $\sim q(x)$.

Q10 If $\sim r(x)$ then $\sim p(x)$ and $\sim q(x)$.

Q11 $(\forall \epsilon > 0)(\exists N)(\forall n \geq N)(|L - a_n| \leq \epsilon)$

negation: $(\exists \epsilon > 0)(\forall N)(\exists n \geq N)(|L - a_n| > \epsilon)$.

claim: $a_n = (-1)^n$ does not have a limit.

proof: for any number L, at least one of $|L-1|, |L+1| \geq \frac{1}{2}$.

choose $\epsilon = \frac{1}{3}$, given N, consider $n = N$ and $n = N+1$, gives $|L-1|, |L+1|$, at least one of these is $\geq \frac{1}{2} > \frac{1}{3}$ as required □.

Q12 Thm $\sqrt{2}$ irrational.

Proof suppose $\sqrt{2} = a/b$ in lowest terms. Then $2 = a^2/b^2$, $2b^2 = a^2 \Rightarrow 2 \mid a^2$ $\Rightarrow 2 \mid a$, but then $a = 2c$ for some $c \in \mathbb{Z}$. So $2b^2 = (2c)^2 = 4c^2 \Rightarrow b^2 = 2c^2$

$\Rightarrow b$ even #. $\square$.

Thm $\sqrt{3}$ irrational.

Proof claim: if $3 | a^2$ then $3 | a$. Proof (contrapositive) suppose not, then $a = 3k+1$ or $3k+2$. Then $a^2 = 9k^2+6k+1$ or $9k^2+12k+4$ $\leftarrow$ neither divisible by 3. $\square$.

Suppose $\sqrt{3} = a/b$ in lowest terms, then $3 = a^2/b^2$ so $3b^2 = a^2 \Rightarrow 3 | a^2$ claim $\Rightarrow 3 | a$ so $a = 3c$ then $3b^2 = (3c)^2 = 9c^2 \Rightarrow b^2 = 3c$ claim $\Rightarrow 3 | b$ # a, b in lowest terms $\square$.

Thm $\sqrt{6}$ irrational.

Proof suppose $\sqrt{6} = a/b$ in lowest terms. Then $6 = a^2/b^2 \Rightarrow 6b^2 = a^2 \Rightarrow a^2$ even $\Rightarrow a$ even $a = 2c$ for some $c \in \mathbb{Z}$. Then $6b^2 = (2c)^2 = 4c^2 \Rightarrow 3b^2 = 2c^2 \Rightarrow 2 | 3b^2$ $\Rightarrow 2 | b^2 \Rightarrow b^2$ even $\Rightarrow b$ even # lowest terms. $\square$.

Thm $\sqrt{2}+\sqrt{3}$ irrational.

Proof consider $(\sqrt{2}+\sqrt{3}) = a/b$ then $(\sqrt{2}+\sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = a^2/b^2$

but then $\underbrace{2\sqrt{6}}_{\text{irrational}} = \underbrace{a^2/b^2 - 5}_{\text{rational}}$ # $\square$.

Q13 Thm $1.2 + 2.3 + 3.4 + \cdots + n.(n+1) = \frac{1}{3} n(n+1)(n+2)$

Proof (induction) Base case $n=1$ $1.2 = \frac{1}{3}.1.2.3 = 2$ ✓.

Induction step. assume $p(n)$, consider $\underbrace{1.2 + 2.3 + \cdots + n(n+1)}_{\text{by } p(n) \,=\, \frac{1}{3}n(n+1)(n+2)} + (n+1)(n+2)$

$= \frac{1}{3} n(n+1)(n+2) + (n+1)(n+2) = (n+1)(n+2)\left[\frac{1}{3}n+1\right] = \frac{1}{3}(n+1)(n+2)(n+3)$ ✓ $\square$

Q14 Thm $n^2 \leq 2^n$ for $n \geq 5$

Proof (induction) Base case $n=5$: $25 \leq 32$ ✓.

Induction step assume $p(n)$: $n^2 \leq 2^n$ as

for $n \geq 1$ $2 + \frac{1}{n} \leq 3$ so for $n \geq 5$ $2 + \frac{1}{n} \leq n \Rightarrow n(2+\frac{1}{n}) \leq n^2$

$\Rightarrow 2n+1 \leq n^2 \Rightarrow n^2+2n+1 \leq 2n^2 \Rightarrow \frac{(n+1)^2}{n^2} \leq 2$.

Therefore $n^2 . \frac{(n+1)^2}{n^2} \leq 2.2^n \Rightarrow (n+1)^2 \leq 2^{n+1}$ $\square$.

Q15 Thm $4^n+5^n+6^n$ divisible by 15 for n odd.

Proof Base case $n=1$: $4+5+6=15$ ✓.

Induction step assume $p(n): 15 \mid 4^n+5^n+6^n$ and consider $4^{n+2}+5^{n+2}+6^{n+2}$.

$$= 4^2 \cdot 4^n + 5^2 \cdot 5^n + 6^2 \cdot 6^n = 4^2 \underbrace{\left(4^n+5^n+6^n\right)}_{\text{divisible by 15 by } p(n)} + \underbrace{9 \cdot 5^n}_{\text{divisible by 15}} + \underbrace{20 \cdot 6^n}_{\text{divisible by 15}}. \quad \square.$$

Q16 Thm $x_1 = 1$, $x_{n+1} = \sqrt{1+2x_n}$, then $x_n \leq 4$ for all n.

Proof (induction) Base case $n=1$: $1 \leq 4$ ✓.

Induction step assume $p(n)$: $x_n \leq 4$. Then $2x_n \leq 8$, $\Rightarrow$

$1+2x_n \leq 9 \Rightarrow \sqrt{1+2x_n} \leq 3 \Rightarrow x_{n+1} \leq 4$. $\square$.

Q17 Thm $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} \leq 2\sqrt{n}$.

Proof (induction). Base case $n=1$ $1 \leq 2$. ✓

Induction step : assume $p(n)$: $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} \leq 2\sqrt{n}$.

$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}} \leq 2\sqrt{n}+\frac{1}{\sqrt{n+1}}$ by $p(n)$.

consider : $0 \leq 1$

$4n^2+4n \leq 4n^2+4n+1$

$4n(n+1) \leq (2n+1)^2$.

$2\sqrt{n(n+1)} \leq 2n+1$

$2\sqrt{n(n+1)}+1 \leq 2(n+1)$

$2\sqrt{n}+\frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1} \Rightarrow 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1} \Rightarrow p(n+1)$ $\square$.

Q18 ~~Thm~~ $x \sim y$ if $x-y$ is irrational not an equivalence relation

check : not reflexive $x \not\sim x$ as $x-x=0$ rational.

Q19 reflexive: pick $A = 1, B = 0$, then $f(x) \leq f(x)$ ✓ so $f \sim f$. 5

symmetric: no, choose $f(x) = 0$, $g(x) = x$, then $f \sim g$ as $0 \leq 0x + 1$ but $g \nsim f$ as for any A, B, $x \nleq A \cdot 0 + B$ as for any A, B choose $B > x + 1$.

Q20 reflexive: choose id map $A \to A$ gives bijection so $A \sim A$.

symmetric if $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ is a bijection so $A \sim B \Rightarrow B \sim A$

transitive: If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection so $A \sim B$ and $B \sim C \Rightarrow A \sim C$.

Therefore $\sim$ is an equivalence relation. Equivalence classes are sets w/same cardinality

Q21 not equivalence relation as not reflexive, e.g. $f(x) = 1$, $f'(x) = 0 \neq f$ so $1 \nsim 1$. Doesn't define a function as not all functions differentiable.