<u>check</u> : boundaries $\mapsto$ boundaries

i.e. $\alpha = \partial\beta \Rightarrow f_\# \alpha = \partial \beta'$

$$\begin{tikzcd} \beta \arrow[r, mapsto, "\partial"] \arrow[d, mapsto] & \alpha \arrow[d, mapsto] \\ f_\# \beta \arrow[r, mapsto] & f_\# \alpha . \end{tikzcd}$$

$$\underbrace{\partial f_\# \beta}_{\beta'} = f_\# \alpha . \quad \checkmark$$

so $f_\#$ induces $f_* : H_n(X) \to H_n(Y)$.

<u>key facts</u> ① $(fg)_* = f_* g_*$

② $\mathbb{1}_* = \mathbb{1}$

<u>Thm</u> If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphisms $f_* = g_* : H_n(X) \to H_n(Y)$.

<u>Corollary</u> $f : X \to Y$ homotopy equivalent $\Rightarrow f_* : H_n(X) \to H_n(Y)$ isomorphism.

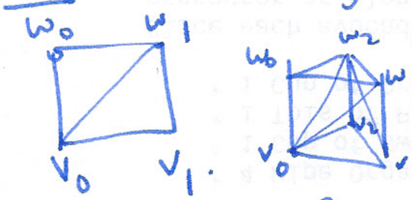
i.e., if X contractible, i.e. $X \simeq \{pt\}$ then $H_n(X) = 0$ if $n \geq 1$
$H_0(X) \cong \mathbb{Z}$.

<u>Proof</u> (of corollary) $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ $\quad gf \simeq id_X$, $\quad fg \simeq id_Y$

$X \to Y \to X$.

$$\begin{tikzcd} H_n(X) \arrow[r, "f_*"'] \arrow[rr, bend right, "id_{H_n(X)}"'] & H_n(Y) \arrow[r, "g_*"'] \arrow[rr, bend left, "id_{H_n(Y)}"] & H_n(X) \arrow[r, "f_*"] & H_n(Y) \end{tikzcd} \quad \square$$

<u>Proof</u> (of Thm). key step: can divide $\Delta^n \times I$ into $(n+1)$-simplices.



$\Delta^1 \times I \quad \Delta^2 \times I$

$\Delta^n \times \{1\} = [w_0, \dots, w_n]$

$\Delta^n \times \{0\} = [v_0, \dots, v_n]$

<u>Fact</u> : the $(n+1)$-simplices : $[v_0, v_1, \dots v_i, w_i, \dots, w_n]$ form a simplicial structure on $\Delta^n \times I$.

let $F : X \times I \to Y$ be a homotopy from f to g

define $P : C_n(X) \to C_{n+1}(Y)$

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \underbrace{F \circ (\sigma \times \mathbb{1})}_{\Delta^n \times I \to X \times I \to Y.} \Big|_{[v_0, \dots, v_i, w_i, \dots w_n]}$$

claim: $\underbrace{\partial P}_{\substack{\text{boundary of} \\ \Delta^n \times I}} = \underbrace{g_\#}_{\text{top}} - \underbrace{f_\#}_{\text{bottom}} - \underbrace{P\partial}_{\text{sides}}$.

check:
$$\partial P(\sigma) = \sum_{j \le i} (-1)^i (-1)^j F\circ(\sigma \times \mathbb{1})\Big|_{[v_0,\dots,\hat{v}_j,\dots v_i, w_i,\dots w_n]}$$
$$+ \sum_{j \ge i} (-1)^i (-1)^{j+1} F\circ(\sigma \times \mathbb{1})\Big|_{[v_0,\dots,v_i, w_i,\dots,\hat{w}_j,\dots w_n]}$$

interior faces $i=j$ cancel out, except for: $F\circ(\sigma\times \mathbb{1})\big|_{[\hat{v}_0, w_0,\dots,w_n]} = g_\#(\sigma)$

and $-F\circ(\sigma\times \mathbb{1})\big|_{[v_0,\dots,v_n,\hat{w}_n]} = f_\#(\sigma)$

terms with $i \neq j$ are exactly $-P\partial(\sigma)$, as

$$P\partial(\sigma) = \sum_{i<j} (-1)^i(-1)^j F\circ(\sigma\times \mathbb{1})\Big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots w_n]}$$
$$+ \sum_{i>j} (-1)^{i-1}(-1)^j F\circ(\sigma\times \mathbb{1})\Big|_{[v_0,\dots\hat{v}_j,\dots v_i,w_i,\dots w_n]}$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_n(X) \arrow[r] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[r] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, "P"'] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$\partial P = f_\# - g_\# - P\partial$

$\partial P + P\partial = f_\# - g_\#$.

cycle: $\alpha \in C_n(X)$ so $\partial\alpha = 0$

$$\downarrow$$

$$g_\#(\alpha) - f_\#(\alpha) = \underbrace{\partial P(\alpha)}_{\text{boundary!}} - P\underbrace{\partial(\alpha)}_{0}$$

so $[g_\#(\alpha)] \sim [f_\#(\alpha)]$ in $H_n(X)$, i.e. $g_*([\alpha]) = f_*([\alpha])$ $\square$.

Def$^{\underline{n}}$ $P: C_n(X) \to C_{n+1}(Y)$ is a chain homotopy ~~iff~~ between $f_\#$ and $g_\#$ if $g_\# - f_\# = \partial P + P\partial$.

Fact: this also works for reduced homology.

Exact sequences and excision

Def$^{\underline{n}}$ Let $\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$ be a sequence of homomorphism between abelian group. The sequence is exact if $\mathrm{im}(\alpha_{n+1}) = \ker(\alpha_n)$ for all n. (equivalent to: A_n is a chain complex with trivial homology).

Remarks

$0 \to A \xrightarrow{\alpha} B$ exact iff α injective

$A \xrightarrow{\alpha} B \to 0$ exact iff α surjective

$0 \to A \xrightarrow{\alpha} B \to 0$ exact iff α isomorphism.

short exact sequence: $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ α inj. β surj.

β gives iso $C \cong B/\mathrm{im}\,\alpha$

pair of spaces (X, A) gives rise to $X, A, X/A$.

Q: is $H_n(X/A) = H_n(X)/H_n(A)$? A: no, but they are related by an exact sequence.

mnemonic: $A \overset{i}{\hookrightarrow} X \xrightarrow{q} X/A$.

Th$^{\underline{m}}$ (X, A) ~~space~~ good pair: $A \neq \phi$, closed, deformation retract of an open set in X.

Then there is an exact sequence:

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X/A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Fact (X, A) A sub CW-complex of CW-complex X, then (X, A) is a good pair.

Corollary $\tilde{H}_n(S^n) \cong \mathbb{Z}$, $\tilde{H}_i(S^n) = 0$ $i \neq n$.

Proof $(X, A) = (D^n, S^{n-1})$, so $X/A = D^n/\partial D^n = S^n$

$$\cdots \to \tilde{H}_k(S^{n-1}) \xrightarrow{i_*} \underset{0}{\tilde{H}_k(D^n)} \xrightarrow{q} \tilde{H}_k(S^n) \xrightarrow{\partial} \cdots$$

$H_k(S^n) \cong H_{k-1}(S^{n-1})$

induction $\square$.