so $\sum \sigma_i = \sum \gamma_i \sigma_i \gamma_i^{-1} \sim \gamma_1\sigma_1\gamma_1^{-1} \cdots \gamma_n \sigma_n \gamma_n^{-1}$ single loop in $\pi_1(X, x_0)$.

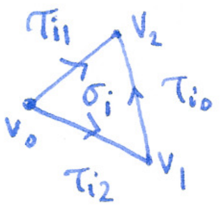
therefore $\exists$ loop $\alpha \in \pi_1(X, x_0)$ s.t. $h(\alpha) \simeq \sum \sigma_i$.

· check $\ker(h) = [\pi_1(X), \pi_1(X)]$

$H_1(X)$ abelian, so $[\pi_1(X), \pi_1(X)] \leq \ker(h)$.

want to show, if $[f] \in \pi_1 X$ s.t. $[f] \in \ker(h)$, then $[f] \in [\pi_1(X), \pi_1(X)]$

$[f] \in \ker(h) \Rightarrow [f]$ bounds a 2-chain $\sum n_i \sigma_i$, wlog each $n_i = \pm 1$.

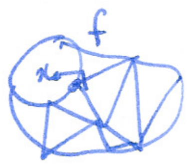


$$f = \partial\left(\sum n_i \sigma_i\right) = \sum n_i \partial(\sigma_i) = \sum_{i,j} (-1)^j n_i \tau_{ij}.$$

so we can pair ~~off~~ all of the edges, except one corresponding to f.

we can identify the paired edges to form a 2-complex K, $\sigma: K \to X$.

we can homotope σ s.t. each vertex gets homotoped to x_0.



e.g. take maximal tree T in 1-skeleton containing x_0.

use: homotopy extension property

<u>Defn</u> (X, A) has the homotopy extension property if every map $X \times \{0\} \cup A \times I \to Y$ can be extended to a map $X \times I \to Y$.

<u>Propn</u> If (X, A) is a CW-pair then (X, A) has the homotopy extension property $\square$.

so in $\pi_1(X)_{ab}$: $[f] = \sum_{i,j} (-1)^j n_i [\tau_{ij}] = \sum n_i \partial \sigma_i$

where $[\partial \sigma_i] = [\tau_{i0} - \tau_{i1} + \tau_{i2}]$. σ_i gives a null homotopy of

$\tau_{i0} - \tau_{i1} + \tau_{i2}$ (as loops!). $\Rightarrow [f] = 0$ in $\pi_1(X)_{ab}$. $\square$.

<u>Fact</u> we can match edges in pairs to form a surface (not just 2-complex), boundary of surface is a commutator. <u>Exercise</u> $f = \partial(\sum n_i \sigma_i) \overset{n_i = \pm 1}{\Rightarrow}$ surface is orientable.

Homotopy invariance

Thm Homotopy equivalent spaces have isomorphic fundamental groups.

In fact: $f : X \to Y$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for each n, and $\cancel{X \simeq Y}$ $f \simeq g \Rightarrow f_* = g_*$ as homomorphisms.

Proof. $f : X \to Y$ determines $f_\# : C_n(X) \to C_n(Y)$

$$(\sigma : \Delta^n \to X) \mapsto f\sigma : \Delta^n \to Y.$$

extend linearly: $\sum n_i \sigma_i \mapsto \sum n_i f\sigma_i$

note:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, "f_\#"] & C_n(X) \arrow[r] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

$f_\# \partial = \partial f_\#$.

check:

$$f_\# \partial(\sigma) = f_\# \left(\sum_{i=0}^{n} (-1)^n \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right)$$

$$= \sum_{i=0}^{n} (-1) f\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} = \partial(f\sigma).$$

Def^{n} $f_\# : C_n(X) \to C_n(Y)$ is a <u>chain map</u> if $\partial f_\# = f_\# \partial$.

claim: $f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[d] \\
C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y)
\end{tikzcd}$$

check: cycles $\mapsto$ cycles.

i.e. $\partial \alpha = 0 \Rightarrow \partial f_\# \alpha = 0$

$$\begin{tikzcd}
\alpha \arrow[r, mapsto, "\partial"] \arrow[d, mapsto] & 0 \arrow[d, mapsto, "f_\#"] \\
f_\# \alpha \arrow[r, mapsto, "\partial"'] & 0
\end{tikzcd}$$

$\partial f_\# \alpha = 0$ ✓