surjective : take any $\sigma_1 : \Delta^0 \to X$.

<u>claim</u> $\ker \epsilon = \operatorname{im} \partial_1$

spose $\sigma : \underset{[v_0, v_1]}{\Delta^1} \to X$ $\quad \partial_1 \sigma = \sigma|_{[v_1]} - \sigma|_{[v_0]}$, so $\epsilon(\partial_1 \sigma) = 1 - 1 = 0$

so $\operatorname{im}(\partial_1) \subset \ker(\epsilon)$

now suppose $\sum_{i=1}^{k} n_i \sigma_i \in \ker(\epsilon)$, i.e. $\sum_{i=1}^{k} n_i = 0$

let $x_0 \in X$ be a basepoint, as X path connected, can choose path $\tau_i : \Delta^1 \to X$ with $\tau_i(v_0) = x_0$ and $\tau_i(v_1) = \sigma_i(pt)$.

consider $\partial\left(\sum n_i \tau_i\right) = \sum n_i \partial \tau_i = \sum n_i \left(\tau_i|_{[v_1]} - \tau_i|_{[v_0]}\right)$

$= \sum n_i \sigma_i - \sum n_i x_0 = \sum n_i \sigma_i$, so $\ker \epsilon \subseteq \operatorname{im}(\partial_1)$. $\square$.

<u>Propn</u> If $X = \{pt\}$ then $H_n(X) = 0 \quad n > 0$
$H_0(X) \cong \mathbb{Z}$.

<u>Proof</u> $C_n(X) \cong \mathbb{Z}$ generated by $\sigma_n : \Delta^n \to \{pt\}$

so
$$\begin{array}{ccccccccccccc} \cdots \to & C_{n+1}(X) & \to & C_n(X) & \to & C_{n-1}(X) & \to \cdots \to & C_2(X) & \to & C_1(X) & \to 0 \\ & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & \end{array}$$

$$\partial \sigma_n = \sum (-1)^n \sigma_n |_{[v_0, \dots, \hat{v}_i, \dots, v_n]} = \sum_{i=0}^{n} (-1)^n \sigma_{n-1} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sigma_{n-1} & \text{if } n \text{ even} \end{cases}$$

$$\left.\begin{array}{l} \cdots \to C_3 \to C_2 \to C_1 \to C_0 \to 0 \\ \cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \end{array}\right\} \begin{array}{l} H_n(X) = 0 \quad n \geq 1 \\ H_0(X) \cong \mathbb{Z}. \end{array}$$
$\square$.

<u>Defn</u> Reduced homology $\tilde{H}_n(X)$: replace $C_0(X) \to 0$
with $C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$

so $H_n(X) = \tilde{H}_n(X)$ for all $n \geq 1$, $\tilde{H}_0(X) = 0$ if X connected.

59

<u>Fact</u> $H_1(X) = ab(\pi_1 X)$ a loop $f : I \to X$ is also a 1-chain $f : \Delta^1 \to X$, in fact a 1-cycle as $\partial f = f|_{[v_1]} - f|_{[v_0]} = 0$

this gives a map $h : \pi_1(X, x_0) \to H_1(X)$

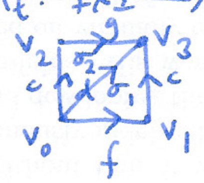
<u>Thm</u> $h : \pi_1(X, x_0) \to H_1(X)$ is surjective, and $\ker(h) =$ commutator subgroup

so $ab(\pi_1(X, x_0)) = \pi_1(X, x_0) / \text{commutator subgp} \cong H_1(X)$. ($X$ path connected)

<u>Proof</u> • <u>check h is well defined</u>: notation: $f \simeq g$ homotopic

$f \sim g$ homologous (~~$h(f) \sim h(g)$~~).

$f \simeq g \Rightarrow \exists$ homotopy $H_t : I \times I \to X$



$c : \Delta^1 \to x_0$ constant loop.

$H_t = \sigma_1 - \sigma_2 \in C_2(X)$

$\partial H_t = f + c - d - (c + g - d) = f - g \in C_1(X)$. so $f \simeq g \Rightarrow f \sim g$.

• <u>note</u> $c : \Delta^1 \to X$ constant path satisfies <u>$c \sim 0$</u>

<u>check</u>: $c_2 : \Delta^2 \to x_0$ has boundary $c_2|_{[v_1, v_2]} - c_2|_{[v_0, v_2]} + c_2|_{[v_0, v_1]}$

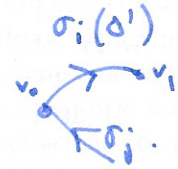
$= c - c + c = c$.

so $c \in \text{im}(\partial_2)$ so $c \sim 0$ in $H_1(X)$.

• <u>check h surjective</u> (X path connected)

a 1-cycle in $C_1(X)$ is $\sum_{i=1}^{k} n_i \sigma_i$, can relabel so $\sum_i \sigma_i$ (i.e. $2\sigma_i = \sigma_i + \sigma_i$)

suppose some $\sigma_i : \Delta^1 \to X$ is not a loop

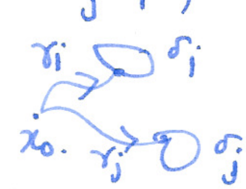


but $\partial \sum_i \sigma_i = 0$

$\Rightarrow \exists\, \sigma_j$ s.t. $\sigma_j(v_1) = \sigma_i(v_0)$

now $\sigma_j \cdot \sigma_i \sim \sigma_i + \sigma_j$, so we can replace $\sigma_i + \sigma_j$ with $\sigma_j \sigma_i$,

continue, reducing $\# \sigma_i$ until each σ_i is a loop



X path connected, choose paths γ_i from x_0 to $\sigma_i(v_0)$

then $\sigma_i \sim \sigma_i + \gamma_i - \gamma_i \sim \gamma_i \sigma_i \gamma_i^{-1} \leftarrow$ loop based at x_0.